السنة 1 بكالوريا علوم رياضية	نهاية دالة عددية	رياضيات النجاح

تمرين 1 : احسب النهايات التالية :

$\lim_{x\mapsto+\infty}\left(\sqrt{9x^2+x+1}-2x\right)$	$\lim_{x\mapsto-\infty}\left(\sqrt{9x^2+x+1}+3x\right)$
$\lim_{x\mapsto+\infty}\left(x^2-\sqrt{x+2}\right)$	$\lim_{x\mapsto\infty}\left(\sqrt{1+2x^3}-\sqrt{x^3+x+1}\right)$
$m\in IR$ حيث $\lim_{x\mapsto-\infty}\left(\sqrt{4x^2+3x+7}-mx\right)$	$\lim_{x\mapsto+\infty}\left(\frac{x-\sqrt{x^2+x+1}}{x^2-\sqrt{x^4-1}}\right)$
$\lim_{x\mapsto-\infty}\left(\frac{x-\sqrt{x^2+x+1}}{x^2-\sqrt{x^4-1}}\right)$	$\lim_{x\mapsto-\infty}\left(\frac{x+1-\sqrt{1-x}}{x^2-\sqrt{x^2+2}}\right)$

تمرين 2 : احسب النهايات التالية :

$\lim_{x\mapsto-1}\left(\frac{\sqrt{1-3x}-2}{x+1}\right)$	$\lim_{x\mapsto2^+}\left(\frac{\sqrt{x^2-4}}{x-2}\right)$
$\lim_{x\mapsto-2^-}\left(\frac{\sqrt{x^2-4}}{x+2}\right)$	$\lim_{x\mapsto1}\left(\frac{x^2+x-2}{x^3+4x^2-8x+3}\right)$
$\lim_{x\mapsto-1}\left(\frac{\sqrt{2+x}+\sqrt{3-x}-3}{x+1}\right)$	$\lim_{x\mapsto0^+}\left(\frac{x+\sqrt{x}}{\sqrt{x^2+x}-x}\right)$

تمرين 3 :

$\lim_{x\mapsto0}\left(\frac{\sqrt{3}\cos x-\sin^2 x-\sqrt{3}}{x^2}\right)$	$\lim_{x\mapsto0}\left(\frac{1}{\sqrt{3}\cos x-\sin^2 x-\sqrt{3}}\right)$
$\lim_{x\mapsto\frac{\pi}{4}}\left(\frac{\tan x-1}{2\cos x-\sqrt{2}}\right)$	$\lim_{x\mapsto\frac{\pi}{6}}\left(\frac{\cos x-\sqrt{3}\sin x}{6x-\pi}\right)$

تمرين 4 : نعتبر الدالة المعرفة كما يلي : $f(x)=x^2\sin\left(\frac{1}{x^2}\right)$

احسب $\lim_{x\mapsto0}f(x)$ ثم $\lim_{x\mapsto+\infty}f(x)$

تمرين 1 :

$$\lim_{x\mapsto+\infty}\left(\sqrt{9x^2+x+1}-2x\right)=\lim_{x\mapsto+\infty}\left(\sqrt{x^2\left(9+\frac{1}{x}+\frac{1}{x^2}\right)}-2x\right)=\lim_{x\mapsto+\infty}\left(x\sqrt{9+\frac{1}{x}+\frac{1}{x^2}}-2x\right)$$

$$\lim_{x\mapsto+\infty}\left(\sqrt{9x^2+x+1}-2x\right)=\lim_{x\mapsto+\infty}x\left(\sqrt{9+\frac{1}{x}+\frac{1}{x^2}}-2\right)=+\infty$$

(لأن : $\lim_{x\mapsto+\infty}\sqrt{9+\frac{1}{x}+\frac{1}{x^2}}-2=3-2=1$ و $\lim_{x\mapsto+\infty}x=+\infty$)

$$\lim_{x\mapsto-\infty}\left(\sqrt{9x^2+x+1}+3x\right)=\lim_{x\mapsto-\infty}\frac{9x^2+x+1-9x^2}{\sqrt{9x^2+x+1}-3x}=\lim_{x\mapsto-\infty}\frac{x+1}{\sqrt{x^2\left(9+\frac{1}{x}+\frac{1}{x^2}\right)}-3x}$$

$$\lim_{x\mapsto-\infty}\left(\sqrt{9x^2+x+1}+3x\right)=\lim_{x\mapsto-\infty}\frac{x\left(1+\frac{1}{x}\right)}{-x\sqrt{9+\frac{1}{x}+\frac{1}{x^2}}-3x}=\lim_{x\mapsto-\infty}\frac{1+\frac{1}{x}}{-\sqrt{9+\frac{1}{x}+\frac{1}{x^2}}-3}=\frac{1}{-3-3}=\frac{-1}{6}$$

- انتبه أنه إذا كان : $x<0$ فإن : $\sqrt{x^2}=|-x|=-x$
- الطريقة السابقة لن تجدي في هذه النهاية لأننا سنحصل على شكل غير محدد
- المثالان متشابهان لكن الطريقة تختلف و السبب المعاملات الموجودة في كل نهاية، لذلك نأمل أن تدرك متى نستعمل المرافق و متى نعمل بأكبر أس.

$$\lim_{x\mapsto+\infty}\left(x^2-\sqrt{x+2}\right)=\lim_{x\mapsto+\infty}\left(x^2-\sqrt{x^2\left(\frac{1}{x}+\frac{2}{x}\right)}\right)=\lim_{x\mapsto+\infty}\left(x^2-x\sqrt{\frac{1}{x}+\frac{2}{x}}\right)=\lim_{x\mapsto+\infty}x\left(x-\sqrt{\frac{1}{x}+\frac{2}{x}}\right)=+\infty$$

(لأن : $\lim_{x\mapsto+\infty}x-\sqrt{\frac{1}{x}+\frac{2}{x}}=+\infty$ و $\lim_{x\mapsto+\infty}x=+\infty$)

$$\lim_{x\mapsto+\infty}\sqrt{1+2x^3}-\sqrt{x^3+x+1}=\lim_{x\mapsto+\infty}\sqrt{x^3}\sqrt{\frac{1}{x^3}+2}-\sqrt{x^3}\sqrt{1+\frac{1}{x^2}+\frac{1}{x^3}}$$

$$\lim_{x\mapsto+\infty}\sqrt{1+2x^3}-\sqrt{x^3+x+1}=\lim_{x\mapsto+\infty}\sqrt{x^3}\left(\sqrt{\frac{1}{x^3}+2}-\sqrt{1+\frac{1}{x^2}+\frac{1}{x^3}}\right)=+\infty$$

(لأن : $\lim_{x\mapsto+\infty}\sqrt{\frac{1}{x^3}+2}-\sqrt{1+\frac{1}{x^2}+\frac{1}{x^3}}=\sqrt{2}-1>0$ و $\lim_{x\mapsto+\infty}\sqrt{x^3}=+\infty$)

- إذا كان $m>0$ فإن : $\lim_{x\mapsto-\infty}\left(\sqrt{4x^2+3x+7}-mx\right)=+\infty$ ($\lim_{x\mapsto-\infty}-mx=+\infty$)
- إذا كان $m=0$ فإن : $\lim_{x\mapsto-\infty}\left(\sqrt{4x^2+3x+7}-mx\right)=\lim_{x\mapsto-\infty}\left(\sqrt{4x^2+3x+7}\right)+\infty$
- إذا كان $-2<m<0$ فإن :

$$\lim_{x\mapsto-\infty}\left(\sqrt{4x^2+3x+7}-mx\right)=\lim_{x\mapsto-\infty}\left(\sqrt{x^2\left(4+\frac{3}{x}+\frac{7}{x^2}\right)}-mx\right)=\lim_{x\mapsto-\infty}\left(-x\sqrt{4+\frac{3}{x}+\frac{7}{x^2}}-mx\right)$$

$$\lim_{x\mapsto-\infty}\left(\sqrt{4x^2+3x+7}-mx\right)=\lim_{x\mapsto-\infty}-x\left(\sqrt{4+\frac{3}{x}+\frac{7}{x^2}}+m\right)=+\infty \qquad (+\infty\times(2+m)>0)$$

- إذا كان $m=-2$ فإن :

$$\lim_{x\mapsto-\infty}\left(\sqrt{4x^2+3x+7}-mx\right)=\lim_{x\mapsto-\infty}\sqrt{4x^2+3x+7}+2x=\lim_{x\mapsto-\infty}\frac{3x+7}{\sqrt{4x^2+3x+7}-2x}$$

$$\lim_{x\mapsto-\infty}\left(\sqrt{4x^2+3x+7}-mx\right)=\lim_{x\mapsto-\infty}\frac{x\left(3+\frac{7}{x}\right)}{-x\sqrt{4+\frac{3}{x}+\frac{7}{x^2}}-2x}=\lim_{x\mapsto-\infty}\frac{3+\frac{7}{x}}{-\sqrt{4+\frac{3}{x}+\frac{7}{x^2}}-2}=\frac{-3}{4}$$

إذا كان $m<-2$ فإن :

$$\lim_{x\mapsto-\infty}\left(\sqrt{4x^2+3x+7}-mx\right)=\lim_{x\mapsto-\infty}-x\left(\sqrt{4+\frac{3}{x}+\frac{7}{x^2}}+m\right)=-\infty \qquad (+\infty\times(2+m)<0)$$

سؤال يوضح بالتفاصيل متى نستعمل طريقة النهاية الأولى و متى نستعمل المرافق

$$\lim_{x\mapsto+\infty}\left(\frac{x-\sqrt{x^2+x+1}}{x^2-\sqrt{x^4-1}}\right)=\lim_{x\mapsto+\infty}\frac{\left(x^2-\left(x^2+x+1\right)\right)\left(x^2+\sqrt{x^4-1}\right)}{\left(x^4-\left(x^4-1\right)\right)\left(x+\sqrt{x^2+x+1}\right)}=\lim_{x\mapsto+\infty}\frac{-(x+1)\left(x^2+x^2\sqrt{1-\frac{1}{x^4}}\right)}{\left(x+x\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}\right)}$$

$$\lim_{x\mapsto+\infty}\left(\frac{x-\sqrt{x^2+x+1}}{x^2-\sqrt{x^4-1}}\right)=\lim_{x\mapsto+\infty}\frac{-x(x+1)\left(1+\sqrt{1-\frac{1}{x^4}}\right)}{1+\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}}=-\infty$$

$$\lim_{x\mapsto-\infty}\left(\frac{x-\sqrt{x^2+x+1}}{x^2-\sqrt{x^4-1}}\right)=\lim_{x\mapsto-\infty}\frac{\left(x-\sqrt{x^2+x+1}\right)\left(x^2+\sqrt{x^4-1}\right)}{\left(x^4-\left(x^4-1\right)\right)}=\lim_{x\mapsto-\infty}\left(x-\sqrt{x^2+x+1}\right)\left(x^2+x^2\sqrt{1-\frac{1}{x^4}}\right)$$

$$\lim_{x\mapsto-\infty}\left(\frac{x-\sqrt{x^2+x+1}}{x^2-\sqrt{x^4-1}}\right)=\lim_{x\mapsto-\infty}x^2\left(x-\sqrt{x^2+x+1}\right)\left(1+\sqrt{1-\frac{1}{x^4}}\right)=-\infty$$

(لأن : $(-\infty+-\infty)$ $\lim_{x\mapsto-\infty}x-\sqrt{x^2+x+1}=-\infty$ و $\lim_{x\mapsto-\infty}x^2=+\infty$ و $\lim_{x\mapsto-\infty}1+\sqrt{1-\frac{1}{x^4}}=1$)

$$\lim_{x\mapsto-\infty}\left(\frac{x+1-\sqrt{1-x}}{x^2-\sqrt{x^2+2}}\right)=\lim_{x\mapsto-\infty}\frac{x+1-\sqrt{x^2\left(\frac{1}{x^2}-\frac{1}{x}\right)}}{x^2-\sqrt{x^2\left(1+\frac{2}{x^2}\right)}}=\lim_{x\mapsto-\infty}\frac{x+1+x\sqrt{\frac{1}{x^2}-\frac{1}{x}}}{x^2+x\sqrt{1+\frac{2}{x^2}}}$$

$$=\lim_{x\mapsto-\infty}\frac{x\left(1+\frac{1}{x}+\sqrt{\frac{1}{x^2}-\frac{1}{x}}\right)}{x\left(x+\sqrt{1+\frac{2}{x^2}}\right)}=\lim_{x\mapsto-\infty}\frac{1+\frac{1}{x}+\sqrt{\frac{1}{x^2}-\frac{1}{x}}}{x+\sqrt{1+\frac{2}{x^2}}}=0$$

(لأن : $\lim_{x\mapsto+\infty}x+\sqrt{1+\frac{2}{x^2}}=-\infty$ و $\lim_{x\mapsto+\infty}1+\frac{1}{x}+\sqrt{\frac{1}{x^2}-\frac{1}{x}}=1$)

تمرين 2 :

$$\lim_{x\mapsto-1}\left(\frac{\sqrt{1-3x}-2}{x+1}\right)=\lim_{x\mapsto-1}\frac{1-3x-4}{(x+1)\left(\sqrt{1-3x}+2\right)}=\lim_{x\mapsto-1}\frac{-3}{\sqrt{1-3x}+2}=\frac{-3}{4}$$

$$\lim_{x\mapsto2^+}\frac{\sqrt{x^2-4}}{x-2}=\lim_{x\mapsto2^+}\sqrt{\frac{(x-2)(x+2)}{(x-2)^2}}=\lim_{x\mapsto2^+}\sqrt{\frac{x+2}{x-2}}=+\infty$$

$$\lim_{x\mapsto -2^-}\frac{\sqrt{x^2-4}}{x+2}=\lim_{x\mapsto 2^-}-\frac{\sqrt{x^2-4}}{-(x+2)}=\lim_{x\mapsto 2^+}-\sqrt{\frac{(x-2)(x+2)}{(x+2)^2}}=\lim_{x\mapsto 2^+}-\sqrt{\frac{x-2}{x+2}}=-\infty$$

$$\lim_{x\mapsto 1}\left(\frac{x^2+x-2}{x^3+4x^2-8x+3}\right)=\lim_{x\mapsto 1}\frac{(x-1)(x+2)}{(x-1)(x^2+5x-3)}=1$$

استعملنا القسمة الإقليدية على $(x-1)$ للتعميل في البسط و المقام

$$\lim_{x\mapsto -1}\left(\frac{\sqrt{2+x}+\sqrt{3-x}-3}{x+1}\right)=\lim_{x\mapsto -1}\left(\frac{\sqrt{2+x}-1}{x+1}+\frac{\sqrt{3-x}-2}{x+1}\right)$$

$$=\lim_{x\mapsto -1}\left(\frac{x+1}{(x+1)(\sqrt{2+x}+1)}+\frac{-1-x}{(x+1)(\sqrt{3-x}+2)}\right)$$

$$=\lim_{x\mapsto -1}\left(\frac{1}{\sqrt{2+x}+1}+\frac{-1}{\sqrt{3-x}+2}\right)$$

$$\lim_{x\mapsto -1}\left(\frac{\sqrt{2+x}+\sqrt{3-x}-3}{x+1}\right)=\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$$

$$\lim_{x\mapsto 0^+}\left(\frac{x+\sqrt{x}}{\sqrt{x^2+x}-x}\right)=\lim_{x\mapsto 0^+}\left(\frac{\sqrt{x}^2+\sqrt{x}}{\sqrt{x(x+1)}-x}\right)=\lim_{x\mapsto 0^+}\frac{\sqrt{x}(\sqrt{x}+1)}{\sqrt{x}\sqrt{x+1}-\sqrt{x}^2}=\lim_{x\mapsto 0^+}\frac{\sqrt{x}(\sqrt{x}+1)}{\sqrt{x}(\sqrt{x+1}-\sqrt{x})}$$

$$\lim_{x\mapsto 0^+}\left(\frac{x+\sqrt{x}}{\sqrt{x^2+x}-x}\right)=\lim_{x\mapsto 0^+}\frac{\sqrt{x}+1}{\sqrt{x+1}-\sqrt{x}}=1$$

تمرين 3 :

$$\lim_{x\mapsto 0}\left(\frac{\sqrt{3}\cos x-\sin^2 x-\sqrt{3}}{x^2}\right)=\lim_{x\mapsto 0}\left(-\sqrt{3}\left(\frac{1-\cos(x)}{x^2}\right)-\left(\frac{\sin(x)}{x}\right)^2\right)=\frac{-\sqrt{3}}{2}-1$$

$$\lim_{x\mapsto 0}\left(\frac{1}{\sqrt{3}\cos x-\sin^2 x-\sqrt{3}}\right)=\lim_{x\mapsto 0}\left(\frac{1}{\sqrt{3}(\cos x-1)-\sin^2 x}\right)=\lim_{x\mapsto 0}\left(\frac{-1}{\sqrt{3}(1-\cos(x))+\sin^2 x}\right)$$

$$\lim_{x\mapsto 0}\left(\frac{1}{\sqrt{3}\cos x-\sin^2 x-\sqrt{3}}\right)=\lim_{x\mapsto 0}\left(\frac{-\frac{1}{x^2}}{\sqrt{3}\left(\frac{1-\cos(x)}{x^2}\right)+\left(\frac{\sin x}{x}\right)^2}\right)=-\infty$$

للتذكير : $\lim_{x\mapsto 0}\frac{\sin x}{x}=1$ و $\lim_{x\mapsto 0}\frac{1-\cos x}{x}=1$

لدينا :

$$\lim_{x\mapsto\frac{\pi}{4}}\left(\frac{\tan x-1}{2\cos x-\sqrt{2}}\right)=\lim_{x\mapsto\frac{\pi}{4}}\left(\frac{\tan x-\tan\left(\frac{\pi}{4}\right)}{2\cos x-\sqrt{2}}\right)=\lim_{x\mapsto\frac{\pi}{4}}\left(\frac{\left(1+\tan(x)\tan\left(\frac{\pi}{4}\right)\right)\tan\left(x-\frac{\pi}{4}\right)}{2\cos x-\sqrt{2}}\right)$$

الآن نضع $t=x-\frac{\pi}{4}$ فنجد :

$$\lim_{x\mapsto\frac{\pi}{4}}\left(\frac{\tan x-1}{2\cos x-\sqrt{2}}\right)=\lim_{t\mapsto0}\left(\frac{\left(1+\tan\left(t+\frac{\pi}{4}\right)\right)\tan(t)}{2\cos\left(t+\frac{\pi}{4}\right)-\sqrt{2}}\right)=\lim_{t\mapsto0}\left(\frac{\left(1+\tan\left(t+\frac{\pi}{4}\right)\right)\tan(t)}{2\left(\cos(t)\frac{\sqrt{2}}{2}-\sin(t)\frac{\sqrt{2}}{2}\right)-\sqrt{2}}\right)$$

$$=\lim_{t\mapsto0}\left(\frac{\left(1+\tan\left(t+\frac{\pi}{4}\right)\right)\tan(t)}{-\sqrt{2}(1-\cos(t))-\sqrt{2}\sin(t)}\right)=\lim_{t\mapsto0}\left(\frac{\left(1+\tan\left(t+\frac{\pi}{4}\right)\right)\frac{\tan(t)}{t}}{-\sqrt{2}\,t\frac{(1-\cos(t))}{t^2}-\sqrt{2}\frac{\sin(t)}{t}}\right)$$

$$\lim_{x\mapsto\frac{\pi}{4}}\left(\frac{\tan x-1}{2\cos x-\sqrt{2}}\right)=\frac{(1+1)\times1}{-\sqrt{2}\times0\times\frac{1}{2}-\sqrt{2}\times1}=-\sqrt{2}$$

لاحظ أن استعمال الخاصية $\tan(a-b)=\frac{\tan(a)-\tan(b)}{1+\tan(a)\tan(b)}$

بالشكل $\tan(a)-\tan(b)=(1+\tan(a)\tan(b))\tan(a-b)$ أفضل من استعمال تغيير المتغير من البداية.

$$\lim_{x\mapsto\frac{\pi}{6}}\left(\frac{\cos x-\sqrt{3}\sin x}{6x-\pi}\right)=\lim_{x\mapsto\frac{\pi}{6}}\left(\frac{2\left(\frac{1}{2}\cos x-\frac{\sqrt{3}}{2}\sin x\right)}{6\left(x-\frac{\pi}{6}\right)}\right)=\lim_{x\mapsto\frac{\pi}{6}}\left(\frac{1}{3}\frac{\left(\sin\left(\frac{\pi}{6}\right)\cos x-\cos\left(\frac{\pi}{6}\right)\sin x\right)}{\left(x-\frac{\pi}{6}\right)}\right)$$

$$\lim_{x\mapsto\frac{\pi}{6}}\left(\frac{\cos x-\sqrt{3}\sin x}{6x-\pi}\right)=\lim_{x\mapsto\frac{\pi}{6}}\left(\frac{1}{3}\frac{\sin\left(\frac{\pi}{6}-x\right)}{\left(x-\frac{\pi}{6}\right)}\right)=\lim_{x\mapsto\frac{\pi}{6}}\left(\frac{-1}{3}\frac{\sin\left(x-\frac{\pi}{6}\right)}{\left(x-\frac{\pi}{6}\right)}\right)=\lim_{t\mapsto0}\left(\frac{-1}{3}\frac{\sin(t)}{t}\right)=\frac{-1}{3}$$

(قمنا بتغيير المتغير x وذلك بوضع $t=x-\frac{\pi}{6}$)، كما يمكن إجراء تغيير المتغير منذ البداية.

تمرين 4 : $f(x)=x^2\sin\left(\frac{1}{x^2}\right)$

لدينا : $-1\leq\sin\left(\frac{1}{x^2}\right)\leq1$ $\forall x\in IR^*$ منه : $-x^2\leq f(x)\leq x^2$ $\forall x\in IR^*$

بما أن : $\lim_{x\mapsto0}x^2=0$ و $\lim_{x\mapsto0}-x^2=0$ فإن : $\lim_{x\mapsto0}f(x)=0$

نضع : $t=\frac{1}{x^2}$ إذن: $\lim_{x\mapsto+\infty}f(x)=\lim_{x\mapsto+\infty}x^2\sin\left(\frac{1}{x^2}\right)=\lim_{t\mapsto0}\frac{1}{t}\sin(t)=1$

السنة 1 بكالوريا علوم رياضية	نهاية دالة عددية	سلسلة 2

تمرين 1 : احسب النهايات التالية :

$\lim_{x \mapsto 1} \frac{\lvert x-1\rvert + x^2 - 1}{x^2 + x - 2}$	$\lim_{x \mapsto +\infty} \frac{\lvert x-10\rvert + x^2}{2x^2 + 5}$
$\lim_{x \mapsto 2} \frac{\lvert x-1\rvert + x^3 - 9}{x^2 - 4}$	$\lim_{x \mapsto 1} \left(\frac{\sqrt{x+3} + \sqrt{x} - 3}{x-1}\right)$
$\lim_{x \mapsto 1} \frac{\sin(\pi x)}{x-1}$	$\lim_{x \mapsto 0} \frac{1 - \cos x \cos 2x}{x^2}$
$\lim_{x \mapsto 1} \frac{x^3\sqrt{x+3} - 2}{x-1}$	$\lim_{x \mapsto \frac{\pi}{4}} \frac{x - \frac{\pi}{4}\tan^2 x}{x - \frac{\pi}{4}}$

تمرين 2 : احسب النهايات التالية :

$\lim_{x \mapsto +\infty} \frac{E(2x)}{x}$	$\lim_{x \mapsto 0} \frac{E(2x)}{x}$
$\lim_{x \mapsto 1} E(x) + x$	$\lim_{x \mapsto 1} E(2x) + \sqrt{x}$
$\lim_{x \mapsto +\infty} \frac{E(2x)}{E(x)}$	$\lim_{x \mapsto 1} E(x^2) - (E(x))^2$

تمرين 3 : احسب النهايات التالية :

$\lim_{x \mapsto 0} \frac{\sqrt{\lvert\cos 3x - \cos x\rvert}}{x}$	$\lim_{x \mapsto +\infty} \sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x}$
$\lim_{x \mapsto -\infty} x\sqrt{x^2+1} + x^2$	$\lim_{x \mapsto \frac{\pi}{4}} \left(\frac{\sqrt{1-\cos(x)} - \sqrt{1-\sin(x)}}{1-\tan(x)}\right)$

تمرين 4 :

1) احسب النهاية $\lim_{x \mapsto a} \frac{x\sqrt{x} - a\sqrt{a}}{\sqrt{x} - \sqrt{a}}$ ($a \in [0;+\infty[$)

2) احسب النهاية $\lim_{x \mapsto 1} \frac{x^{2015} - 1 - 2015(x-1)}{(x-1)^2}$

السنة 1 بكالوريا علوم رياضية	نهاية دالة عددية حلول مقترحة	سلسلة 2

تمرين 1 :

$$\lim_{\substack{x\mapsto 1\\ x>1}} \frac{|x-1|+x^2-1}{x^2+x-2} = \lim_{\substack{x\mapsto 1\\ x>1}} \frac{x-1+x^2-1}{x^2+x-2} = \lim_{\substack{x\mapsto 1\\ x>1}} \frac{x^2+x-2}{x^2+x-2} = 1$$

$$\lim_{\substack{x\mapsto 1\\ x<1}} \frac{|x-1|+x^2-1}{x^2+x-2} = \lim_{\substack{x\mapsto 1\\ x<1}} \frac{1-x+x^2-1}{x^2+x-2} = \lim_{\substack{x\mapsto 1\\ x<1}} \frac{x^2-x}{x^2+x-2} = \lim_{\substack{x\mapsto 1\\ x<1}} \frac{x(x-1)}{(x-1)(x+2)} = \lim_{\substack{x\mapsto 1\\ x<1}} \frac{x}{x+2} = \frac{1}{3}$$

لدينا : $\forall x \in [10+\infty[\quad x-10 \geq 0$ منه :

$$\lim_{x\mapsto +\infty} \frac{|x-10|+x^2}{2x^2+5} = \lim_{x\mapsto +\infty} \frac{x-10+x^2}{2x^2+5} = \lim_{x\mapsto +\infty} \frac{x^2}{2x^2} = \frac{1}{2}$$

- يمكننا دائما اعتبار $x \in [a;+\infty[$ عند حساب نهاية في $+\infty$
- يمكننا دائما اعتبار $x \in]-\infty;a]$ عند حساب نهاية في $-\infty$

لدينا : $\forall x \in [1;3] \quad x-1 \geq 0$ منه :

$$\lim_{x\mapsto 2} \frac{|x-1|+x^3-9}{x^2-4} = \lim_{x\mapsto 2} \frac{x-1+x^3-9}{x^2-4} = \lim_{x\mapsto 2} \frac{x^3-8+x-2}{x^2-4}$$

$$\lim_{x\mapsto 2} \frac{|x-1|+x^3-9}{x^2-4} = \lim_{x\mapsto 2} \frac{(x-2)(x^2+2x+4+1)}{(x-2)(x+2)} = \lim_{x\mapsto 2} \frac{x^2+2x+5}{x+2} = \frac{13}{4}$$

- يمكننا دائما اعتبار $x \in [a,b]$ حيث $x_0 \in [a,b]$ عند حساب نهاية في x_0

$$\lim_{x\mapsto 1} \left(\frac{\sqrt{x+3}+\sqrt{x}-3}{x-1}\right) = \lim_{x\mapsto 1} \left(\frac{\sqrt{x+3}-2}{x-1} + \frac{\sqrt{x}-1}{x-1}\right) = \lim_{x\mapsto 1} \left(\frac{x+3-4}{(x-1)(\sqrt{x+3}+2)} + \frac{x-1}{(x-1)(\sqrt{x}+1)}\right)$$

$$\lim_{x\mapsto 1} \left(\frac{\sqrt{x+3}+\sqrt{x}-3}{x-1}\right) = \lim_{x\mapsto 1} \left(\frac{1}{\sqrt{x+3}+2} + \frac{1}{\sqrt{x}+1}\right) = \frac{1}{4}+\frac{1}{2} = \frac{3}{4}$$

(نضع $t = x-1$)

$$\lim_{x\mapsto 1} \frac{\sin(\pi x)}{x-1} = \lim_{t\mapsto 0} \frac{\sin(\pi(t+1))}{t} = \lim_{t\mapsto 0} \frac{\sin(\pi t+\pi)}{t} = \lim_{t\mapsto 0} \pi \frac{\sin(\pi t)}{\pi t} = \pi$$

$$\lim_{x\mapsto 0} \frac{1-\cos x\cos 2x}{x^2} = \lim_{x\mapsto 0} \frac{1-\cos x+\cos x-\cos x\cos 2x}{x^2} = \lim_{x\mapsto 0} \frac{1-\cos x}{x^2} + \cos x\frac{1-\cos 2x}{x^2}$$

$$\lim_{x\mapsto 0} \frac{1-\cos x\cos 2x}{x^2} = \lim_{x\mapsto 0} \frac{1-\cos x}{x^2} + 4\cos x\frac{1-\cos 2x}{(2x)^2} = \frac{1}{2} + 4\times 1\times\frac{1}{2} = \frac{5}{2}$$

$$\lim_{x\mapsto 1} \frac{x^3\sqrt{x+3}-2}{x-1} = \lim_{x\mapsto 1} \frac{x^3(\sqrt{x+3}-2+2)-2}{x-1} = \lim_{x\mapsto 1} \frac{x^3(\sqrt{x+3}-2)}{x-1} + \frac{2x^3-2}{x-1}$$

$$\lim_{x\mapsto 1} \frac{x^3\sqrt{x+3}-2}{x-1} = \lim_{x\mapsto 1} x^3\frac{x+3-4}{(x-1)(\sqrt{x+3}+2)} + 2\frac{(x-1)(x^2+x+1)}{x-1} = \lim_{x\mapsto 1} x^3\frac{1}{\sqrt{x+3}+2} + 2(x^2+x+1)$$

$$\lim_{x\mapsto 1} \frac{x^3\sqrt{x+3}-2}{x-1} = \frac{1}{4}+6 = \frac{25}{4}$$

$$\lim_{x\mapsto\frac{\pi}{4}} \frac{x-\frac{\pi}{4}\tan^2 x}{x-\frac{\pi}{4}} = \lim_{x\mapsto\frac{\pi}{4}} \frac{x-\frac{\pi}{4}+\frac{\pi}{4}-\frac{\pi}{4}\tan^2 x}{x-\frac{\pi}{4}} = \lim_{x\mapsto\frac{\pi}{4}} 1+\frac{\pi}{4}\frac{1-\tan^2 x}{x-\frac{\pi}{4}} = \lim_{x\mapsto\frac{\pi}{4}} 1-\frac{\pi}{4}(\tan x+1)\frac{\tan x-1}{x-\frac{\pi}{4}}$$

$$\lim_{x\mapsto\frac{\pi}{4}} \frac{x-\frac{\pi}{4}\tan^2 x}{x-\frac{\pi}{4}} = \lim_{x\mapsto\frac{\pi}{4}} 1-\frac{\pi}{4}(1+\tan x)\frac{\tan\left(x-\frac{\pi}{4}\right)(1+\tan x)}{x-\frac{\pi}{4}} = \lim_{x\mapsto\frac{\pi}{4}} 1-\frac{\pi}{4}(1+\tan x)^2\frac{\tan\left(x-\frac{\pi}{4}\right)}{x-\frac{\pi}{4}}$$

$$\lim_{x\mapsto\frac{\pi}{4}} \frac{x-\frac{\pi}{4}\tan^2 x}{x-\frac{\pi}{4}} = 1-\frac{\pi}{4}\times 4\times 1 = 1-\pi$$

للتذكير : $\tan(x-y)=\dfrac{\tan x-\tan y}{1+\tan x\tan y}$ ما يعني أن : $\tan x-\tan y=\tan(x-y)(1+\tan x\tan y)$

تمرين 2 :

نعلم أن : $2x-1<E(2x)\le 2x \quad \forall x>0$ منه $\forall x>0 \quad 2-\frac{1}{x}<\frac{E(2x)}{x}\le 2$ ، بما أن :

$$\lim_{x\mapsto+\infty} 2-\frac{1}{x} = \lim_{x\mapsto+\infty} 2 = 2$$

فإن : $\lim_{x\mapsto+\infty}\frac{E(2x)}{x}=2$

لدينا : $x\in\left]0;\frac{1}{2}\right[\Rightarrow 2x\left]0;1\right[\Rightarrow E(2x)=0 \Rightarrow \frac{E(2x)}{x}=0$ منه : $\lim_{\substack{x\mapsto 0\\ x>0}}\frac{E(2x)}{x}=\lim_{\substack{x\mapsto 0\\ x>0}} 0=0$

و لدينا : $x\in\left]-\frac{1}{2};0\right[\Rightarrow 2x\left]-1;0\right[\Rightarrow E(2x)=-1 \Rightarrow \frac{E(2x)}{x}=\frac{-1}{x}$ منه : $\lim_{\substack{x\mapsto 0\\ x<0}}\frac{E(2x)}{x}=\lim_{\substack{x\mapsto 0\\ x<0}}\frac{-1}{x}=+\infty$

تمعن جيدا في الطريقة المتبعة في كل نهاية، فدالة الجزء الصحيح دالة خاصة جدا لذلك نستعمل طرقا خاصة بها.

عند حساب النهاية يمين عدد x_0 يمكنك دائما أن تعتبر $x\in\left]x_0;a\right[$ (أيضا على اليسار $x\in\left]a;x_0\right[$)

لدينا : $x\in\left]1;2\right[\Rightarrow E(x)=1 \Rightarrow E(x)+x=1+x$ منه : $\lim_{\substack{x\mapsto 1\\ x>1}} E(x)+x=\lim_{\substack{x\mapsto 1\\ x>1}} 1+x=2$

و $x\in\left]0;1\right[\Rightarrow E(x)=0 \Rightarrow E(x)+x=x$ منه : $\lim_{\substack{x\mapsto 1\\ x<1}} E(x)+x=\lim_{\substack{x\mapsto 1\\ x<1}} x=1$

لا يمكننا التعويض ببساطة كما نفعل مع جل الدوال الاعتيادية، فدالة الجزء الصحيح نقول عنها في الرياضيات أنها دالة غير متصلة

لدينا : $x\in\left]1;\frac{3}{2}\right[\Rightarrow 2x\in\left]2;3\right[\Rightarrow E(2x)=2 \Rightarrow E(2x)+\sqrt{x}=2+x$

منه : $\lim_{\substack{x\mapsto 1\\ x>1}} E(2x)+\sqrt{x}=\lim_{\substack{x\mapsto 1\\ x>1}} 2+x=3$

و $x\in\left]\frac{1}{2};1\right[\Rightarrow 2x\in\left]1;2\right[\Rightarrow E(2x)=1 \Rightarrow E(2x)+\sqrt{x}=1+x$ منه : $\lim_{\substack{x\mapsto 1\\ x<1}} E(2x)+\sqrt{x}=\lim_{\substack{x\mapsto 1\\ x<1}} 1+x=2$

نعلم أن : $\begin{cases}2x-1<E(2x)\le 2x\\ x-1<E(x)\le x\end{cases} \quad \forall x>0$ منه : $\forall x>1 \quad \begin{cases}2x-1<E(2x)\le 2x\\ \frac{1}{x}<\frac{1}{E(x)}\le\frac{1}{x-1}\end{cases}$

منه : $\forall x>1 \quad \frac{2x-1}{x}<\frac{E(2x)}{E(x)}\le\frac{2x}{x-1}$ وبما أن : $\lim_{x\mapsto+\infty}\frac{2x-1}{x}=\lim_{x\mapsto+\infty}\frac{2x}{x-1}=\lim_{x\mapsto+\infty}\frac{2x}{x}=2$ فإن: $\lim_{x\mapsto+\infty}\frac{E(2x)}{E(x)}=2$

لدينا : $x \in]1;\sqrt{2}[\Rightarrow \begin{cases} x^2 \in]1;2[\\ x \in]1;2[\end{cases} \Rightarrow \begin{cases} E(x^2)=1 \\ E(x)=1 \end{cases} \Rightarrow E(x^2)-(E(x))^2 = 1-1^2 = 0$

منه: $\lim\limits_{\substack{x \mapsto 1 \\ x>1}} E(x^2)-(E(x))^2 = \lim\limits_{\substack{x \mapsto 1 \\ x>1}} 0 = 0$

و لدينا : $x \in]0;1[\Rightarrow \begin{cases} x^2 \in]0;1[\\ x \in]0;1[\end{cases} \Rightarrow \begin{cases} E(x^2)=0 \\ E(x)=0 \end{cases} \Rightarrow E(x^2)-(E(x))^2 = 0-0^2 = 0$

منه: $\lim\limits_{\substack{x \mapsto 1 \\ x<1}} E(x^2)-(E(x))^2 = \lim\limits_{\substack{x \mapsto 1 \\ x<1}} 0 = 0$ ، خلاصة: $\lim\limits_{x \mapsto 1} E(x^2)-(E(x))^2 = 0$

تمرين 3 : احسب النهايات التالية:

$$\lim_{\substack{x \mapsto 0 \\ x>0}} \frac{\sqrt{|\cos 3x - \cos x|}}{x} = \lim_{\substack{x \mapsto 0 \\ x>0}} \sqrt{\frac{|\cos 3x - \cos x|}{x^2}} = \lim_{\substack{x \mapsto 0 \\ x>0}} \sqrt{\left|\frac{\cos 3x - 1}{x^2} + \frac{1-\cos x}{x^2}\right|}$$

$$\lim_{\substack{x \mapsto 0 \\ x>0}} \frac{\sqrt{|\cos 3x - \cos x|}}{x} = \lim_{\substack{x \mapsto 0 \\ x>0}} \sqrt{\left|-9\frac{1-\cos 3x}{(3x)^2} + \frac{1-\cos x}{x^2}\right|} = \sqrt{\left|\frac{-9}{2} + \frac{1}{2}\right|} = 2$$

$$\lim_{\substack{x \mapsto 0 \\ x<0}} \frac{\sqrt{|\cos 3x - \cos x|}}{x} = \lim_{\substack{x \mapsto 0 \\ x<0}} -\sqrt{\frac{|\cos 3x - \cos x|}{x^2}} = -2$$

$\forall x > 0 \;\; x = x^2$ و $\forall x < 0 \;\; x = -x^2$

$$\lim_{x \mapsto +\infty} \sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x} = \lim_{x \mapsto +\infty} \frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}} = \lim_{x \mapsto +\infty} \frac{\sqrt{x\left(1+\frac{\sqrt{x}}{x}\right)}}{\sqrt{x\left(1+\frac{\sqrt{x+\sqrt{x}}}{x}\right)}+\sqrt{x}}$$

$$\lim_{x \mapsto +\infty} \sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x} = \lim_{x \mapsto +\infty} \frac{\sqrt{x}\sqrt{1+\frac{\sqrt{x}}{x}}}{\sqrt{x}\left(\sqrt{1+\sqrt{\frac{x+\sqrt{x}}{x^2}}}+1\right)} = \lim_{x \mapsto +\infty} \frac{\sqrt{1+\frac{1}{\sqrt{x}}}}{\sqrt{1+\sqrt{\frac{1}{x}+\frac{1}{x\sqrt{x}}}}+1} = \frac{1}{2}$$

يمكن وضع $t = x^2$ للتبسيط

$$\lim_{x \mapsto -\infty} x\sqrt{x^2+1}+x^2 = \lim_{t \mapsto +\infty} -t\sqrt{t^2+1}+t^2 = \lim_{t \mapsto +\infty} t^2 - t\sqrt{t^2+1} = \lim_{t \mapsto +\infty} \frac{t^4-t^4-t^2}{t^2+t\sqrt{t^2+1}}$$

(نضع $t = -x$)

$$\lim_{x \mapsto -\infty} x\sqrt{x^2+1}+x^2 = \lim_{t \mapsto +\infty} \frac{-t^2}{t^2+t^2\sqrt{1+\frac{1}{t^2}}} = \lim_{t \mapsto +\infty} \frac{-1}{1+\sqrt{1+\frac{1}{t^2}}} = \frac{-1}{2}$$

$$\lim_{x\mapsto\frac{\pi}{4}}\left(\frac{\sqrt{1-\cos(x)}-\sqrt{1-\sin(x)}}{1-\tan(x)}\right)=\lim_{x\mapsto\frac{\pi}{4}}\frac{\sin(x)-\cos(x)}{(1-\tan(x))\left(\sqrt{1-\cos(x)}+\sqrt{1-\sin(x)}\right)}$$

$$=\lim_{x\mapsto\frac{\pi}{4}}\frac{\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin(x)-\frac{1}{\sqrt{2}}\cos(x)\right)}{\left(\tan\left(\frac{\pi}{4}-x\right)(1+\tan(x))\right)\left(\sqrt{1-\cos(x)}+\sqrt{1-\sin(x)}\right)}$$

$$=\lim_{x\mapsto\frac{\pi}{4}}\frac{\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)\sin(x)-\sin\left(\frac{\pi}{4}\right)\cos(x)\right)}{\left(\tan\left(\frac{\pi}{4}-x\right)(1+\tan(x))\right)\left(\sqrt{1-\cos(x)}+\sqrt{1-\sin(x)}\right)}$$

$$=\lim_{x\mapsto\frac{\pi}{4}}\frac{\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)}{\left(\tan\left(\frac{\pi}{4}-x\right)(1+\tan(x))\right)\left(\sqrt{1-\cos(x)}+\sqrt{1-\sin(x)}\right)}$$

$$=\lim_{x\mapsto\frac{\pi}{4}}\frac{-\sqrt{2}}{(1+\tan(x))\left(\sqrt{1-\cos(x)}+\sqrt{1-\sin(x)}\right)}\left(\frac{\sin\left(x-\frac{\pi}{4}\right)}{x-\frac{\pi}{4}}\right)\left(\frac{x-\frac{\pi}{4}}{\tan\left(x-\frac{\pi}{4}\right)}\right)$$

$$=\frac{-\sqrt{2}}{2\times\left(\sqrt{1-\frac{\sqrt{2}}{2}}+\sqrt{1-\frac{\sqrt{2}}{2}}\right)}\times1\times1=\frac{-\sqrt{2}}{4\sqrt{\frac{2-\sqrt{2}}{2}}}=\frac{-2}{4\sqrt{2-\sqrt{2}}}$$

$$\lim_{x\mapsto\frac{\pi}{4}}\left(\frac{\sqrt{1-\cos(x)}-\sqrt{1-\sin(x)}}{1-\tan(x)}\right)=\frac{-1}{2}\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2}}=\frac{-\sqrt{4+2\sqrt{2}}}{4}$$

تمرين 4 :

$$\lim_{x\mapsto a}\frac{x\sqrt{x}-a\sqrt{a}}{\sqrt{x}-\sqrt{a}}=\lim_{x\mapsto a}\frac{\left(\sqrt{x}\right)^3-\left(\sqrt{a}\right)^3}{\sqrt{x}-\sqrt{a}}=\lim_{x\mapsto a}\frac{\left(\sqrt{x}-\sqrt{a}\right)\left(\sqrt{x}^2+\sqrt{x}\sqrt{a}+\sqrt{a}^2\right)}{\sqrt{x}-\sqrt{a}}=\lim_{x\mapsto a}x+\sqrt{a\,x}+a=3a$$

$$\lim_{x\mapsto1}\frac{x^{2015}-1-2015(x-1)}{(x-1)^2}=\lim_{x\mapsto1}\frac{(x-1)\left(x^{2014}+x^{2013}+\ldots+x+1\right)-2015(x-1)}{(x-1)^2}$$

$$=\lim_{x\mapsto1}\frac{x^{2014}+x^{2013}+\ldots+x+1-2015}{x-1}$$

$$=\lim_{x\mapsto1}\frac{x^{2014}+x^{2013}+\ldots+x-2014}{x-1}=\lim_{x\mapsto1}\frac{x^{2014}-1}{x-1}+\frac{x^{2013}-1}{x-1}+\ldots+\frac{x-1}{x-1}$$

$$=\lim_{x\mapsto1}\left(x^{2013}+x^{2012}+\ldots+x+1\right)+\left(x^{2012}+x^{2011}+\ldots+x+1\right)+\ldots+1$$

$$\lim_{x\mapsto1}\frac{x^{2015}-1-2015(x-1)}{(x-1)^2}=2014+2013+\ldots+1=\frac{2014\times2015}{2}=2029105$$