☑ تمارين محلولة: الدوال الأصلية وحساب التكامل

1 التمرين 1.

حدد الدوال الأصلية للدوال التالية :

(1 $f(x) = x^3 - 3x + 1$

(2 $f(x) = \frac{1}{x^2} + 2x$

(3 $f(x) = \frac{1}{x} + x$

(4 $f(x) = \sqrt{x + 2}$

(5 $f(x) = (x + 1)\sqrt{x + 1}$

(6 $f(x) = (x + 2) \quad x + 1$

(7 $f(x) = \frac{x + 2}{\sqrt{x} + 1}$

(8 $f(x) = \frac{x}{x^2 + 1}$

(9 $f(x) = x(x^2 + 1)^3$

(10 $f(x) = \frac{2x + 1}{(x^2 + x + 1)^2}$

✓ الحل ▴

(1 $f(x) = x^3 - 3x + 1$

$$\boxed{F(x) = \frac{1}{4}x^4 - \frac{3}{2}x^2 + x + c \quad ; \quad c \in \mathbb{R}}$$

(2 $f(x) = \frac{1}{x^2} + 2x$

$$\boxed{F(x) = -\frac{1}{x} + \frac{2}{2}x^2 + c \quad ; \quad c \in \mathbb{R}}$$

(3 $f(x) = \frac{1}{x} + x$

لدينا $(\ x)' = \frac{1}{2\ x}$ و $x = x^{\frac{1}{2}}$

إذن $F(x) = 2\ x + \frac{1}{\frac{1}{2} + 1} x^{\frac{1}{2} + 1} + c$

$$= 2\ x + \frac{2}{3} x^{\frac{3}{2}} + c$$

إذن $\boxed{F(x) = 2\ x + \frac{2}{3} x\ \ x + c \quad ; \quad c \in \mathbb{R}}$

(4 $f(x) = \sqrt{x} + 1 = (x + 2)^{\frac{1}{2}} \times 1$

$$F(x) = \frac{1}{\frac{1}{2} + 1}(x + 2)^{\frac{1}{2} + 1} + c$$

$$\boxed{F(x) = \frac{2}{3}(x + 2)^{\frac{3}{2}} + c \ ; \quad c \in \mathbb{R}}$$

(5 $f(x) = (x + 1) \quad x + 1 = (x + 1)^{\frac{3}{2}} (x + 1)($

$$F(x) = \frac{1}{\frac{3}{2}+1}(x+1)^{\frac{3}{2}+1} + c$$

إذن $\boxed{F(x) = \frac{2}{5}(x+1)^2 .\ x+1+c \ ;\ c \in \mathbb{R}}$

6)

$$f(x) = (x+2)\ x+1 = (x+1)\ x+1 + \ x+1$$

$$\boxed{F(x) = \frac{2}{5}(x+1)^2\ x+1 + \frac{2}{3}(x+1)\ x+1+c \ ;\ c \in \mathbb{R}}$$

7) $$f(x) = \frac{x+2}{x+1} = \frac{x+1}{x+1} + \frac{1}{x+1}$$

$$= \ x+1 + 2 . \frac{(x+1)'}{2\ x+1}$$

$$\boxed{F(x) = \frac{2}{3}(x+1)\ x+1+2\ x+1+c ; c \in \mathbb{R}}$$

8) $$f(x) = \frac{x}{\sqrt{x^2+1}} = \frac{(x^2+1)'}{2\ x^2+1}$$

$$\boxed{F(x) = \ x^2+1+c \ ;\ c \in \mathbb{R}}$$

9) $$f(x) = x(x^2+1)^3 = \frac{1}{2}(x^2+1)' . (x^2+1)^3$$

$$F(x) = \frac{1}{2} . \frac{1}{4}(x^2+1)^4 + c$$

أي $\boxed{F(x) = \frac{1}{8}(x^2+1)^4 + c \ ;\ c \in \mathbb{R}}$

10)

$$f(x) = \frac{2x+1}{(x^2+x+1)^2} = (x^2+x+1)'(x^2+x+1)^{-2}$$

$$F(x) = \frac{1}{-2+1}(x^2+x+1)^{-2+1} + c$$

$$\boxed{F(x) = \frac{-1}{x^2+x+1} + c \ ;\ c \in \mathbb{R}}$$

2 التمرين 2.

حدد الدوال الأصلية لكل من الدوال التالية :

1) $f_1(x) = \frac{1}{x+2}$ 2) $f_2(x) = \frac{1}{2x-1}$ 3) $f_3(x) = \frac{x}{x^2+2}$

4) $f_4(x) = \frac{2x+1}{x^2+x+2}$ 5) $f_5(x) = \frac{x+1}{x^2+2x+3}$ 6) $f_6(x) = \frac{x^2+1}{x^3+3x+2}$

✓ الحل ▴

نضع F دالة أصلية لـ f

1) $f_1(x) = \frac{1}{x+2} \ ;\ x \in \mathbb{R} - \{-2\}$

لدينا : $f_1(x) = \frac{(x+2)'}{x+2}$

إذن $\boxed{F_1(x) = \ln | x+2 | + c}$ $c \in \mathbb{R}$

2) $f_2(x) = \frac{1}{2x-1} \ ;\ x \in \mathbb{R} - \left\{\frac{1}{2}\right\}$

$$f_2(x) = \frac{1}{2}\,\frac{(2x-1)'}{2x-1}$$

إذن $\boxed{F_2(x) = \frac{1}{2}\ln|2x-1| + c}$ $c \in \mathbb{R}$

(3 $f_3(x) = \dfrac{x}{x^2+2}$; $x \in \mathbb{R}$

$$f_3(x) = \frac{1}{2}\,\frac{(x^2+2)'}{x^2+2}$$

إذن $\boxed{F_3(x) = \frac{1}{2}\ln(x^2+2) + c}$ $c \in \mathbb{R}$

(4 $f_4(x) = \dfrac{2x+1}{x^2+x+2}$; $x \in \mathbb{R}$

$$f_4(x) = \frac{(x^2+x+2)'}{x^2+x+2}$$

إذن $\boxed{F_4(x) = \ln(x^2+x+2) + c}$ $c \in \mathbb{R}$

(5 $f_5(x) = \dfrac{x+1}{x^2+2x+3}$; $x \in \mathbb{R}$

$$f_5(x) = \frac{1}{2}\,\frac{(x^2+2x+3)'}{x^2+2x+3}$$

(6 $\boxed{F_5(x) = \frac{1}{2}\ln(x^2+2x+3) + c}$ $c \in \mathbb{R}$

$$f_6(x) = \frac{x^2+1}{x^3+3x+2}$$

$$f_6(x) = \frac{1}{3}\,\frac{(x^3+3x+2)'}{x^3+3x+2}$$

إذن $\boxed{F_6(x) = \frac{1}{3}\ln|x^3+3x+2| + c}$ $c \in \mathbb{R}$

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التمرين 3.

لتكن f الدالة العددية للمتغير الحقيقي x المعرفة بما يلي : $f(x) = \dfrac{x}{x^2-3x+2}$

1) حدد D حيز تعريف الدالة f.

2) أوجد العددين الحقيقيين a و b حيث : $f(x) = \dfrac{a}{x-1} + \dfrac{b}{x-2}$ $\quad \forall x \in D$

3) حدد الدالة الأصلية F للدالة f على D التي تنعدم في الصفر.

✓ الحل ▴

$$f(x) = \frac{x}{x^2-3x+2}$$

(1
$$\begin{aligned} D &= \{x \in \mathbb{R} \,/\, x^2-3x+2 \neq 0\} \\ &= \mathbb{R} - \{1; 2\} \end{aligned}$$

2) لدينا :

$$\begin{aligned} \frac{a}{x-1} + \frac{b}{x-2} &= \frac{a(x-2) + b(x-1)}{(x-1)(x-2)} \\ &= \frac{(a+b)x - (2a+b)}{x^2-3x+2} \end{aligned}$$

$$\forall x \in D \,;\, f(x) = \frac{a}{x-1} + \frac{b}{x-2} \Leftrightarrow \begin{cases} a+b = 1 \\ -(2a+b) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} b = 1 - a \\ 2a + 1 - a = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} b = 1 - a \\ a + 1 = 0 \end{cases}$$

$\Leftrightarrow a = -1$ و $b = 2$

إذن $\boxed{\forall x \in D \; ; f(x) = \frac{-1}{x-1} + \frac{2}{x-2}}$

3) بما أن : $f(x) = \frac{-1}{x-1} + \frac{2}{x-2}$

فإن : $F(x) = -\ln|x-1| + 2\ln|x-2| + c$

لنحدد c حيث $f(0) = 0$

$$F(0) = 0 \Leftrightarrow \ln 1 + 2\ln 2 + c = 0$$
$$\Leftrightarrow c = -2\ln 2$$

إذن $\boxed{F(x) = 2\ln|x-2| - \ln|x-1| - 2\ln 2}$

4

التمرين 4.

حدد الدوال الأصلية لكل من الدوال التالية :

1) $f_1(x) = \frac{\ln x}{x}$

2) $f_2(x) = \frac{(\ln x)^2}{x}$

3) $f_3(x) = \frac{(\ln x)^3}{x}$

4) $f_4(x) = \frac{1}{x \ln x}$

5) $f_5(x) = \frac{1}{x(\ln x)^2}$

6) $f_6(x) = \frac{1}{x(\ln x)^3}$

✓ الحل ▴

1) $f_1(x) = \frac{\ln x}{x}$

$f_1(x) = \ln x \, . (\ln x)'$

إذن $\boxed{F_1(x) = \frac{1}{2}\ln^2 x + c}$ $c \in \mathbb{R}$

2) $f_2(x) = \frac{1}{x}\ln^2 x$

$f_2(x) = (\ln x)'\ln^2 x$

إذن $\boxed{F_2(x) = \frac{1}{3}\ln^3 x + c}$ $c \in \mathbb{R}$

3) $f_3(x) = \frac{1}{x}\ln^3 x$

$f_3(x) = (\ln x)'\ln^3 x$

إذن $\boxed{F_3(x) = \frac{1}{4}\ln^4 x + c}$ $c \in \mathbb{R}$

4) $f_4(x) = \frac{1}{x \ln x}$

$f_4(x) = \frac{(\ln x)'}{\ln x}$

إذن $\boxed{F_4(x) = \ln|\ln x| + c}$ $c \in \mathbb{R}$

5) $f_5(x) = \frac{1}{x\ln^2 x}$

$f_5(x) = \frac{(\ln x)'}{\ln^2 x}$

إذن $c \in IR$ $\boxed{F_5(x) = -\frac{1}{\ln x} + c}$

6) $f_6(x) = \frac{1}{x \ln^3 x}$

$f_6(x) = \frac{(\ln x)'}{\ln^3 x}$

إذن $c \in IR$ $\boxed{F_6(x) = \frac{-1}{2 \ln^2 x} + c}$

5

التمرين 5.

أحسب التكاملات التالية :

$I_1 = \int_0^2 3\,dx$, $I_2 = \int_1^2 (2x - 1)\,dx$

$I_3 = \int_{-1}^1 (x^2 + x + 1)\,dx$, $I_4 = \int_0^1 (x - 2)(x + 1)\,dx$

$I_5 = \int_0^1 \left(\frac{3}{2}x^3 - \frac{2}{2}x\right)dx$, $I_6 = \int_{-1}^2 \left(x - \frac{1}{2}\right)^2 dx$

✓ الحل ▴

تذكير :

الدالة $x \to \frac{1}{n+1}x^{n+1}$ هي الدالة الأصلية للدالة $x \to x^n$ على IR.

- $I_1 = \int_1^2 3\,dx = [3x]_0^2 = 6$
- $I_2 = \int_1^2 (2x - 1)\,dx = [x^2 - x]_1^2 = 2$
- $I_3 = \int_{-1}^1 (x^2 + x + 1)\,dx = \left[\frac{x^3}{3} + \frac{x^2}{2} + x\right]_{-1}^1 = \frac{8}{3}$
- $I_4 = \int_0^1 (x - 2)(x + 1)\,dx$

$= \int_0^1 (x^2 - x - 2)\,dx$

$= \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_0^1$

$= -\frac{7}{6}$

- $I_5 = \int_0^1 \left(\frac{3}{2}x^3 - \frac{2}{2}x\right)dx$

$= \left[\frac{3}{8}x^4 - \frac{2}{4}x^2\right]_0^1$

$= \frac{3 - 2\ \ 2}{8}$

$I_6 = \int_{-1}^2 \left(x - \frac{1}{2}\right)^2 dx$

$= \int_{-1}^2 \left(x^2 - x + \frac{1}{4}\right)dx$

$= \left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{4}x\right]_{-1}^2$

$= \frac{9}{4}$

6

التمرين 6.

أحسب التكاملات التالية :

$$J_2 = \int_{\frac{1}{2}}^{1} \frac{-1}{x^3}\,dx \quad , \quad J_1 = \int_{1}^{2} \frac{2}{x^2}\,dx$$

$$J_4 = \int_{1}^{e} \frac{1}{x}\,dx \quad , \quad J_3 = \int_{\frac{1}{2}}^{\frac{1}{3}} \left(\frac{3}{x^4} - \frac{1}{x^5}\right)dx$$

$$J_6 = \int_{1}^{2} \left(x^2 - \frac{1}{x^2}\right)^2 dx \quad , \quad J_5 = \int_{1}^{2} \frac{(x+1)^2}{x^4}\,dx$$

✓ الحل ▴

- $$J_1 = \int_{1}^{2} \frac{2}{x^2}\,dx = \left[\frac{-2}{x}\right]_{1}^{2} = 1$$

- $$J_2 = \int_{\frac{1}{2}}^{1} \frac{-1}{x^3}\,dx$$
$$= \left[-1 .\frac{1}{-3+1}\,x^{-3+1}\right]_{\frac{1}{2}}^{1}$$
$$= \left[\frac{1}{2x^2}\right]_{\frac{1}{2}}^{1} = -\frac{3}{2}$$

- $$J_3 = \int_{\frac{1}{2}}^{\frac{1}{3}} \left(\frac{3}{x^4} - \frac{1}{x^5}\right)dx$$
$$= \left[3 .\frac{1}{-4+1}\,x^{-4+1} - \frac{1}{-5+1}\,x^{-5+1}\right]_{\frac{1}{2}}^{\frac{1}{3}}$$
$$= \left[-\frac{1}{x^3} + \frac{1}{4x^4}\right]_{\frac{1}{2}}^{\frac{1}{3}}$$
$$= -\frac{11}{4}$$

- $$J_4 = \int_{1}^{e} \frac{1}{x}\,dx$$
$$= [\ln|x|]_{1}^{e} = 1$$

- $$J_5 = \int_{1}^{2} \frac{(x+1)^2}{x^4}\,dx$$
$$= \int_{1}^{2} \frac{x^2+2x+1}{x^4}\,dx$$
$$= \int_{1}^{2} \left(\frac{1}{x^2} + \frac{2}{x^3} + \frac{1}{x^4}\right)dx$$
$$= \left[-\frac{1}{x} - \frac{1}{x^2} - \frac{1}{3x^3}\right]_{1}^{2}$$
$$= \frac{37}{24}$$

- $$J_6 = \int_{1}^{2} \left(x^2 - \frac{1}{x^2}\right)^2 dx$$
$$= \int_{1}^{2} \left(x^4 - 2 + \frac{1}{x^4}\right)dx$$
$$= \left[\frac{x^5}{5} - 2x - \frac{1}{3x^3}\right]_{1}^{2}$$
$$= \frac{31}{5} + \frac{7}{24} - 2$$

7

التمرين 7.

أحسب التكاملات التالية :

$I_1 = \int_{-1}^{1} (x - 2)^3 \, dx \quad , \quad I_2 = \int_{-1}^{2} \sqrt{x + 2} \, dx$

$I_3 = \int_{1}^{-2} \sqrt{2 - x} \, dx \quad , \quad I_4 = \int_{0}^{4} \sqrt{2x + 1}$

$I_5 = \int_{0}^{4} \frac{dx}{\sqrt{2x + 1}} \quad , \quad I_6 = \int_{1}^{2} \frac{dx}{(2x + 1)^2}$

✓ الحل ▴

نعلم أن $x \mapsto \frac{1}{a} \cdot \frac{1}{r + 1} (ax + b)^{r + 1}$ $(r \in \mathbb{Q} - \{-1\})$ هي دالة

أصلية للدالة $x \longmapsto (ax + b)^r$ على مجال تكون فيه $ax + b > 0$.

- $$I_1 = \int_{-1}^{1} (x - 2)^3 \, dx$$
$$= \left[\frac{1}{4} (x - 2)^4\right]_{1}^{1}$$
$$= -20$$

- $$I_2 = \int_{-1}^{2} x + 2 \, dx$$
$$= \int_{-1}^{2} (x + 2)^{\frac{1}{2}} \, dx$$
$$= \left[\frac{1}{\frac{1}{2} + 1} (x + 2)^{\frac{1}{2} + 1}\right]_{-1}^{2}$$
$$= \left[\frac{2}{3} (x + 2) \quad x + 2\right]_{1}^{2}$$
$$= \frac{14}{3}$$

- $$I_3 = \int_{1}^{-2} \quad 2 - x \, dx$$
$$= \int_{1}^{-2} (2 - x)^{\frac{1}{2}} \, dx$$
$$= \left[\frac{1}{-1} \cdot \frac{1}{\frac{1}{2} + 1} (2 - x)^{\frac{1}{2} + 1}\right]_{1}^{-2}$$
$$= \left[-\frac{2}{3} \cdot (2 - x) \cdot \quad 2 - x\right]_{1}^{-2}$$
$$= -\frac{14}{3}$$

- $$I_4 = \int_{0}^{4} \sqrt{2x + 1} \, dx$$
$$= \int_{0}^{4} (2x + 1)^{\frac{1}{2}} \, dx$$
$$= \left[\frac{1}{2} \cdot \frac{1}{\frac{1}{2} + 1} (2x + 1)^{\frac{1}{2} + 1}\right]_{0}^{4}$$
$$= \left[\frac{1}{3} \cdot (2x + 1) \quad 2x + 1\right]_{0}^{4}$$
$$= \frac{26}{3}$$

- $$I_5 = \int_{0}^{4} \frac{dx}{2x + 1}$$
$$= \int_{0}^{4} (2x + 1)^{-\frac{1}{2}} \, dx$$

$$= \left[\frac{1}{2} \cdot \frac{1}{-\frac{1}{2}+1} (2x+1)^{-\frac{1}{2}+1} \right]_0^4$$

$$= \left[2x+1 \right]_0^4$$

$$= 2$$

●

$$I_6 = \int_1^2 \frac{dx}{(2x+1)^2}$$

$$= \int_1^2 (2x+1)^{-2} dx$$

$$= \left[\frac{1}{2} \cdot \frac{1}{-2+1} (2x+1)^{-2+1} \right]_1^2$$

$$= \left[\frac{-1}{2(2x+1)} \right]_1^2$$

$$= \frac{1}{9}$$

8 التمرين 8.

أحسب التكاملات التالية :

$$I_1 = \int_0^1 (2x+1)(x^2+x+1)^2 dx \quad , \quad I_2 = \int_{-1}^1 \frac{2x+1}{(x^2+x+1)^3} dx$$

$$I_3 = \int_0^1 \frac{2x-1}{x^2-x+1} dx \quad . \quad I_4 = \int_1^2 \frac{x+1}{x^2+2x+3} dx$$

✓ الحل ▴

●

$$I_1 = \int_0^1 (2x+1)(x^2+x+1)^2 dx$$

$$= \int_0^1 (x^2+x+1)'(x^2+x+1)^2 dx$$

$$= \left[\frac{1}{2+1} (x^2+x+1)^{2+1} \right]_0^1$$

$$= \left[\frac{1}{3} (x^2+x+1)^3 \right]_0^1$$

$$= \frac{2}{3}$$

●

$$I_2 = \int_{-1}^1 \frac{2x+1}{(x^2+x+1)^3} dx$$

$$= \int_{-1}^1 (x^2+x+1)'(x^2+x+1)^{-3} dx$$

$$= \left[\frac{1}{-3+1} (x^2+x+1)^{-3+1} \right]_{-1}^1$$

$$= \left[\frac{-1}{2(x^2+x+1)^2} \right]_{-1}^1$$

$$= \frac{4}{9}$$

●

$$I_3 = \int_0^1 \frac{2x-1}{x^2-x+1} dx$$

$$= \int_0^1 (x^2-x+1)'(x^2-x+1)^{-\frac{1}{2}} dx$$

$$= \left[\frac{1}{-\frac{1}{2}+1} (x^2-x+1)^{-\frac{1}{2}+1} \right]_0^1$$

$$= \left[2\ \ x^2 - x + 1\right]_0^1 = 0$$

- $$I_4 = \int_1^2 \frac{x + 1}{x^2 + 2x + 3}\,dx$$

$$= \frac{1}{2}\int_1^2 \frac{(x^2 + 2x + 3)'}{x^2 + 2x + 3}\,dx$$

$$= \frac{1}{2}\left[\ln|x^2 + 2x + 3|\right]_1^2$$

$$= \frac{1}{2}\ln\left(\frac{11}{6}\right)$$

9 التمرين 9.

أحسب التكاملات التالية :

$I_1 = \int_0^1 e^{2x}\,dx$, $I_2 = \int_0^1 e^{3x-1}\,dx$

$I_3 = \int_{-1}^1 x\,e^{x^2}\,dx$, $I_4 = \int_0^1 (x + 1)\,e^{x^2+2x+3}\,dx$

✓ الحل ▴

نعلم أن الدالة $x \longmapsto e^{u(x)}$ هي دالة أصلية للدالة $x \longmapsto u'(x)\,e^{u(x)}$ للدالة على D_u.

- $$I_1 = \int_0^1 e^{2x}\,dx$$

$$= \left[\frac{1}{2}e^{2x}\right]_0^1$$

$$= \frac{1}{2}(e^2 - 1)$$

$$I_2 = \int_0^1 e^{3x-1}\,dx$$

$$= \left[\frac{1}{3}e^{3x-1}\right]_0^1$$

$$= \frac{1}{3}(e^2 - e^{-1})$$

$$I_3 = \int_{-1}^1 x\,e^{x^2}\,dx$$

$$= \frac{1}{2}\int_{-1}^1 (x^2)'\,e^{x^2}\,dx$$

$$= \frac{1}{2}\left[e^{x^2}\right]_{-1}^1$$

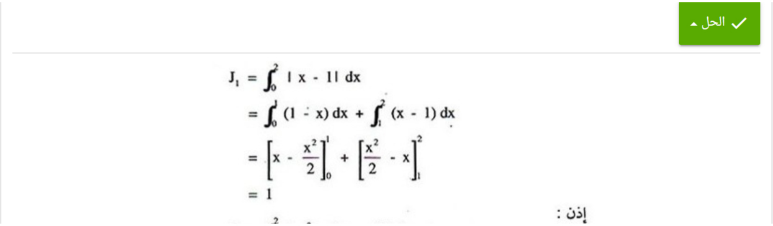
$$= 0$$

$$I_4 = \int_0^1 (x + 1)\,e^{x^2+2x+3}\,dx$$

$$= \frac{1}{2}\int_0^1 (x^2 + 2x + 3)'\,e^{x^2+2x+3}\,dx$$

$$= \frac{1}{2}\left[e^{x^2+2x+3}\right]_0^1$$

$$= \frac{1}{\ }(e^6 - e^3)$$



$$
\begin{aligned}
J_1 &= \int_0^2 |x-1|\,dx \\
&= \int_0^1 (1-x)\,dx + \int_1^2 (x-1)\,dx \\
&= \left[x-\frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2}-x\right]_1^2 \\
&= 1
\end{aligned}
$$

إذن :

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