

لك عدد طبيعي n

$$b_n = \int_0^{\frac{\pi}{2}} t^2 \cos^{2n} t \, dt \quad \text{و} \quad a_n = \int_0^{\frac{\pi}{2}} \cos^n t \, dt$$

(1) أحسب a_0 ; b_0

$$(2) \quad (\forall n \in \mathbb{N}) \quad a_{n+2} = \frac{n+1}{n+2} a_n$$

$$(3) \quad \text{أ-} \quad \text{بيه أنه} \quad t \leq \frac{\pi}{2} \sin t \quad \left(\forall t \in \left[0, \frac{\pi}{2} \right] \right)$$

$$\text{ب- استنتج أنه} \quad 0 < b_n \leq \frac{\pi^2}{4} (a_{2n} - a_{2n+2}) \quad (\forall n \in \mathbb{N})$$

$$\text{ج-} \quad \text{بيه أنه} \quad \lim_{n \rightarrow +\infty} \frac{b_n}{a_{2n}} = 0$$

(4) أ- باستعمال مكاملة بالأجزاء بيه أنه :

$$(\forall n \in \mathbb{N}) \quad a_{2n+2} = (2n+2) \int_0^{\frac{\pi}{2}} t \sin t \cos^{2n+1} t \, dt$$

$$\text{ب- استنتج أنه} \quad (\forall n \in \mathbb{N}) \quad \frac{a_{2n+2}}{n+1} = (2n+1)b_n - (2n+2)b_{n+1}$$

$$\text{ج- استنتج أنه} \quad (\forall n \in \mathbb{N}) \quad 2 \left(\frac{b_n}{a_{2n}} - \frac{b_{n+1}}{a_{2n+2}} \right) = \frac{1}{(n+1)^2}$$

$$(5) \quad \text{أ-} \quad \text{بيه أنه المتالية} \quad U_n = \sum_{k=1}^{k=n} \frac{1}{k^2} \quad \text{متقاربة و أنه نهايتها هي} \quad \frac{\pi^2}{6}$$

$$\text{ب- استنتج أنه} \quad \lim_{n \rightarrow +\infty} \sum_{k=0}^{k=n} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

الجزء (1) ليكن n عدد طبيعي غير منعدم .

$$\text{نعتبر الدالة العددية} \quad g_n \quad \text{المعرفة على} \quad \mathbb{R}^{+*} \quad \text{بما يلي :} \quad g_n(x) = n \ln x - \frac{1}{x}$$

$$(1) \quad \text{أحسب النهايتين} \quad \lim_{x \rightarrow 0^+} g_n(x) \quad ; \quad \lim_{x \rightarrow +\infty} g_n(x)$$

(2) اعط جدول تغيرات الدالة g_n

$$\text{أ-} \quad \text{بيه أنه المعادلة} \quad g_n(x) = 0 \quad \text{تقبل حلا وحيدا} \quad \alpha_n \quad \text{و استنتج إشارة} \quad g_n(x)$$

$$\text{ب-} \quad \text{بيه أنه} \quad 1 < \alpha_n \quad (\forall n \in \mathbb{N}^*) \quad \text{و استنتج أنه} \quad \alpha_n < e^{\frac{1}{n}} \quad (\forall n \in \mathbb{N}^*)$$

$$\text{ج-} \quad \text{استنتج أنه} \quad (\alpha_n)_n \quad \text{متقاربة و حدد نهايتها و بيه أنه} \quad \lim_{n \rightarrow +\infty} n(\alpha_n - 1) = 1$$

الجزء (2) لك عدد طبيعي غير منعدم n ،

$$\text{نعتبر الدالة المعرفة على} \quad \mathbb{R}^+ - \{1\} \quad \text{بما يلي :} \quad x > 0 \quad ; \quad f_n(x) = \frac{e^{nx}}{\ln x} \quad \text{و} \quad f_n(0) = 0$$

$$(1) \quad \text{أحسب النهايات} \quad \lim_{x \rightarrow +\infty} f_n(x) \quad ; \quad \lim_{x \rightarrow 1^+} f_n(x) \quad \text{و} \quad \lim_{x \rightarrow 1^-} f_n(x)$$

$$(2) \quad \text{بيه أنه الدالة} \quad f_n \quad \text{متصلة على يمين النقطة} \quad a = 0$$

$$(3) \quad \text{أدرسه قابلية اشتقاق الدالة} \quad f_n \quad \text{على يمين النقطة} \quad a = 0$$

$$\text{أ-} \quad \text{بيه أنه} \quad (\forall x \in \mathbb{R}^{+*} - \{1\}) \quad f_n'(x) = \frac{e^{nx}}{(\ln x)^2} g_n(x)$$

ب- اعط جدول تغيرات الدالة f_n

$$(4) \quad \text{أدرسه الفرع اللانهائي للمنحنى} \quad (C_n) \quad \text{عند} \quad +\infty$$

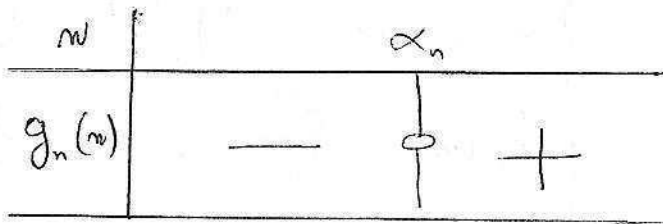
$$(5) \quad \text{أرسم المنحنى} \quad (C_1) \quad \text{(نعطي} \quad \alpha_1 = 1,75 \quad \text{و} \quad f_1(\alpha_1) = 10,2)$$

تصحيح الفرض المحروس رقم 1

الدورة الثانية

على المجال $] \alpha_n, +\infty [$

$$\left. \begin{array}{l} n > \alpha_n \\ g_n \text{ تزايدية} \end{array} \right\} \Rightarrow g_n(n) > g_n(\alpha_n) \\ \Rightarrow g_n(n) > 0$$



③ لدينا $g_n(1) < 0$ و $g_n(\alpha_n) = 0$

$$g_n(1) < g_n(\alpha_n)$$

و g_n تزايدية، $1 < \alpha_n$

$$\alpha_n = \frac{1}{n \cdot \alpha_n} \Leftrightarrow g_n(\alpha_n) = 0 \quad *$$

$$\alpha_n \cdot n > n \Leftrightarrow \alpha_n > 1$$

$$\frac{1}{\alpha_n \cdot n} < \frac{1}{n} \quad \text{و من}$$

$$\alpha_n < e^{\frac{1}{n}} \quad \text{اذن } \alpha_n < \frac{1}{n} \quad \text{من}$$

$$1 < \alpha_n < e^{\frac{1}{n}} \quad *$$

$$\lim_{n \rightarrow +\infty} e^{\frac{1}{n}} = 1 \quad \text{و}$$

وبالتالي

$$\lim_{n \rightarrow +\infty} \alpha_n = \boxed{1}$$

تمرين 01 $g_n(n) = n \cdot \ln(n) - \frac{1}{n}$

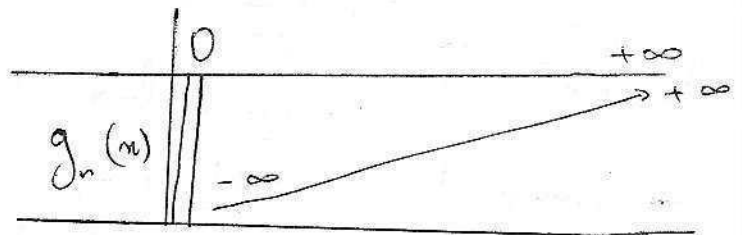
$$\lim_{n \rightarrow +\infty} g_n(n) = \lim_{n \rightarrow +\infty} n \ln(n) - \frac{1}{n}$$

$$\lim_{n \rightarrow +\infty} g_n(n) = \boxed{+\infty}$$

$$\lim_{n \rightarrow 0^+} g_n(n) = \lim_{n \rightarrow 0^+} n \ln(n) - \frac{1}{n}$$

$$\lim_{n \rightarrow 0^+} g_n(n) = \boxed{-\infty}$$

$$g'_n(n) = \frac{n}{n} + \frac{1}{n^2} > 0 \quad \text{②}$$



* بين أن $\exists! \alpha_n \in \mathbb{R}^{+*} / g_n(\alpha_n) = 0$

- الدالة g_n متصلة وتزايدية قطبا على \mathbb{R}^{+*} و $g_n(n)$ تقابل من \mathbb{R}^+ نحو $0 \in \mathbb{R}$

اذن $\exists! \alpha_n \in \mathbb{R}^{+*} / g_n(\alpha_n) = 0$

على المجال $] 0, \alpha_n [$

$$\left. \begin{array}{l} 0 < n < \alpha_n \\ g_n \text{ تزايدية} \end{array} \right\} \Rightarrow g_n(n) < g_n(\alpha_n) \\ \Rightarrow g_n(n) < 0$$

$$\lim_{+\infty} \frac{n}{\ln(n)} = +\infty$$

$$\lim_{+\infty} f_n(n) = \boxed{+\infty} \text{ alog}$$

$$\lim_{0^+} f_n(n) = \lim_{0^+} \frac{e^{nw}}{\ln(n)} \text{ L'Hôpital (2)}$$

$$\lim_{0^+} \frac{e^{nw}}{\ln n} = 0 \Leftrightarrow \begin{cases} \lim_{0^+} e^{nw} = e^0 = 1 \\ \lim_{0^+} \ln n = -\infty \end{cases}$$

$$\lim_{0^+} f_n(n) = f_n(0) \leftarrow$$

0 uyas ite alpsio f_n

$$\lim_{0^+} \frac{f_n(n) - f_n(0)}{n - 0} = \lim_{0^+} \frac{e^{nw}}{n \ln n} \text{ (c)}$$

$$\lim_{0^+} \frac{e^{nw}}{n \ln n} = -\infty \Leftrightarrow \begin{cases} \lim_{0^+} e^{nw} = 1 \\ \lim_{0^+} n \ln n = 0^- \end{cases}$$

C_n g uyas ite alpsio f_n

0 uyas ite alpsio f_n

$$f_n'(n) = \frac{n \cdot e^{nw} \ln(n) - \frac{e^{nw}}{n}}{(\ln(n))^2} \text{ (3)}$$

$$f_n'(n) = \frac{n \cdot e^{nw} \left(n \ln(n) - \frac{1}{n} \right)}{n^2 (\ln(n))^2}$$

$$f_n'(n) = \frac{e^{nw}}{(\ln(n))^2} \cdot g_n(n)$$

$$\lim_{+\infty} n(\alpha_n - 1) = 1 \text{ alog}$$

$$n = \frac{1}{\alpha_n \ln \alpha_n} \Leftrightarrow g_n(\alpha_n) = 0$$

$$\lim_{+\infty} n(\alpha_n - 1) = \lim_{+\infty} \frac{1}{\alpha_n} \cdot \frac{\alpha_n - 1}{\ln \alpha_n} \text{ alog}$$

$$\lim_{+\infty} \alpha_n = 1 \Rightarrow \lim_{+\infty} \frac{\ln \alpha_n}{\alpha_n - 1} = 1$$

$$\lim_{+\infty} \frac{1}{\alpha_n} = 1 \text{ g}$$

$$\lim_{+\infty} n(\alpha_n - 1) = \boxed{1} \text{ alog}$$

$$\begin{cases} f_n(n) = \frac{e^{nw}}{\ln n} \\ f_n(0) = 0 \end{cases} \text{ (II)}$$

f_n ite alpsio (1)

$$\lim_{n \rightarrow 1} f_n(n) = \lim_{n \rightarrow 1} \frac{e^{nw}}{\ln(n)}$$

$$\lim_{n \rightarrow 1} e^{nw} = e \text{ alog}$$

$$\lim_{n \rightarrow 1} \ln(n) = 0 \text{ g}$$

$$n > 1 \Rightarrow \ln(n) > 0$$

$$\Rightarrow \lim_{n \rightarrow 1^+} f_n(n) = +\infty$$

$$n < 1 \Rightarrow \ln(n) < 0$$

$$\lim_{n \rightarrow 1^-} f_n(n) = -\infty$$

$$\lim_{+\infty} \frac{e^{nw}}{\ln(n)} = \lim_{+\infty} \frac{e^{nw}}{nw} \cdot \frac{n}{\ln(n)}$$

$$\lim_{+\infty} \frac{e^{nw}}{nw} = +\infty$$

تصريف 02

$$a_0 = \int_0^{\pi/2} 1 dt = [t]_0^{\pi/2} = \frac{\pi}{2} \quad (1)$$

$$b_0 = \int_0^{\pi/2} t^2 dt = \left[\frac{1}{3} t^3 \right]_0^{\pi/2} = \frac{\pi^3}{24}$$

$$a_{n+2} = \int_0^{\pi/2} \cos^{n+2}(t) dt \quad (2)$$

$$= \int_0^{\pi/2} \cos t \cdot \cos^{n+1} t dt$$

نضع

$$u = \cos^{n+1} t$$

$$v = \cos t$$

$$u' = -(n+1) \cos^n t \sin t$$

$$v' = \sin t$$

$$a_{n+2} = \left[\sin t \cos^{n+1} t \right]_0^{\pi/2} + \int_0^{\pi/2} (n+1) \cos^n t \cdot \sin^2 t dt$$

$$= (1 \times 0 - 1 \times 0) + (n+1) \int_0^{\pi/2} \cos^n t (1 - \cos^2 t) dt$$

$$a_{n+2} = (n+1) \int_0^{\pi/2} \cos^n t - (n+1) \int_0^{\pi/2} \cos^{n+2} t dt$$

$$(n+2) a_{n+2} = (n+1) a_n$$

$$a_{n+2} = \frac{n+1}{n+2} a_n$$

$$0 < b_n < \frac{\pi^2}{4} (a_{2n} - a_{2n+2}) \quad (3)$$

$$0 < t < \frac{\pi}{2} \Rightarrow \sin t > 0$$

$$0 < t^2 < \frac{\pi^2}{4} \sin^2 t$$

$$\forall t \in [0, \frac{\pi}{2}] ; \cos^{2n} t > 0$$

$$f'_n(x) = \frac{e^{nx}}{(\ln(x))^2} g_n(x)$$

$e^{nx} > 0$ و $\ln(x)^2 > 0$ لدينا
 $g_n(x)$ هي التي ندرسها $f'_n(x)$ هي التي ندرسها

x	0	1	a_n	$+\infty$
$f'_n(x)$	—		— 0 +	+
$f_n(x)$	0	—	$f_n(a_n)$	$+\infty$

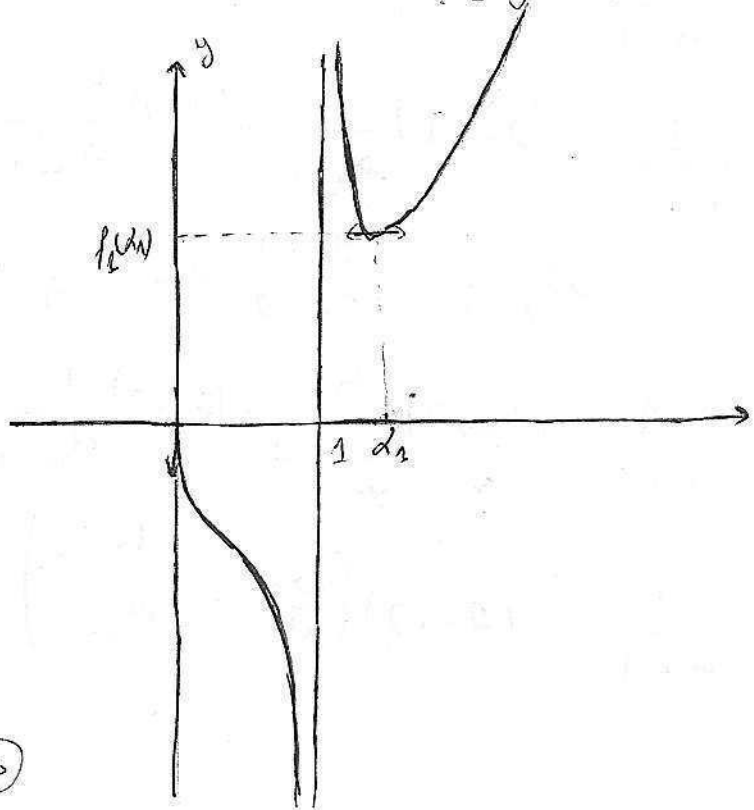
$$\lim_{+\infty} \frac{f_n(x)}{x} = \lim_{+\infty} \frac{e^{nx}}{x \cdot \ln(x)^2} \quad (4)$$

$$= \lim_{+\infty} n^2 \cdot \frac{e^{nx}}{(n \cdot x)^2} \times \frac{x}{\ln x}$$

$$\lim_{+\infty} \frac{e^{nx}}{(nx)^2} = +\infty \quad \text{و} \quad \lim_{+\infty} \frac{x}{\ln x} = +\infty$$

$$\lim_{+\infty} \frac{f_n(x)}{x} = +\infty \quad \Leftarrow$$

C_n يقبل فرع سلوكها في اتجاه
 محور الـ x الرأسي



$$a_{2n+2} = 2 \int_0^{\pi/2} t \sin t \cos^{2n+1} t dt \quad (4)$$

$$J_n = \int_0^{\pi/2} t \sin t \cos^{2n+1} t dt \quad \text{عزيم}$$

$$u = \frac{1}{2} t^2 \quad \Leftrightarrow \begin{cases} u' = t \\ v = \sin t \cos^{2n+1} t \end{cases}$$

$$v'(t) = \cos^{2n+2} t - (2n+1) \cos^{2n} t \sin t$$

$$v'(t) = (2n+2) \cos^{2n+2} t - (2n+1) \cos^{2n} t \sin t$$

$$J_n = \left[\frac{1}{2} t^2 \sin t \cos^{2n+1} t \right]_0^{\pi/2}$$

$$- \frac{1}{2} \int_0^{\pi/2} ((2n+2)t^2 \cos^{2n+2} t - (2n+1)t^2 \cos^{2n} t \sin t) dt$$

$$= \frac{1}{2} \int_0^{\pi/2} (2n+1)t^2 \cos^{2n} t dt - \frac{1}{2} \int_0^{\pi/2} (2n+2)t^2 \cos^{2n+2} t dt$$

$$J_n = \frac{1}{2} (2n+1) b_n - \frac{1}{2} (2n+2) b_{n+1} \quad \text{حيث}$$

$$\frac{a_{2n+2}}{n+1} = (2n+1) b_n - (2n+2) b_{n+1}$$

النتيجة

$$\frac{a_{2n+2}}{n+1} = (2n+1) b_n - (2n+2) b_{n+1}$$

$$\frac{1}{n+1} = (2n+1) \frac{b_n}{a_{2n+2}} - (2n+2) \frac{b_{n+1}}{a_{2n+2}}$$

$$a_{2n+2} = \frac{2n+1}{2n+2} a_n \quad \text{حيث}$$

$$\frac{1}{n+1} = (2n+1) \frac{b_n}{\frac{2n+1}{2n+2} a_n} - (2n+2) \frac{b_{n+1}}{a_{2n+2}}$$

$$\frac{1}{n+1} = (2n+2) \left(\frac{b_n}{a_{2n}} - \frac{b_{n+1}}{a_{2n+2}} \right)$$

$$0 \int_0^{\pi/2} t^2 \cos^{2n} t \int_0^{\pi/2} \frac{\pi^2}{4} \cos^{2n} t \sin^2 t$$

$$0 \int_0^{\pi/2} t^2 \cos^{2n} t \int_0^{\pi/2} \frac{\pi^2}{4} (\cos^{2n} t - \cos^{2n+2} t)$$

$$0 \int_0^{\pi/2} t^2 \cos^{2n} t dt \int_0^{\pi/2} (\cos^{2n} t - \cos^{2n+2} t) dt$$

$$0 \int_0^{\pi/2} b_n \int_0^{\pi/2} \frac{\pi^2}{4} (a_{2n} - a_{2n+2}) \quad \text{حيث}$$

$$0 \int_0^{\pi/2} \frac{b_n}{a_{2n}} \int_0^{\pi/2} \frac{\pi^2}{4} \left(1 - \frac{a_{2n+2}}{a_{2n}} \right)$$

$$a_{2n+2} = \frac{2n+1}{2n+2} a_{2n}$$

$$0 \int_0^{\pi/2} \frac{b_n}{a_{2n}} \int_0^{\pi/2} \frac{\pi^2}{4} \cdot \frac{1}{2n+2}$$

$$\lim_{n \rightarrow \infty} \frac{\pi^2}{4} \cdot \frac{1}{2n+2} = 0 \quad \text{حيث}$$

$$\lim_{n \rightarrow \infty} \frac{b_n}{a_{2n}} = 0 \quad \text{حيث}$$

$$a_{2n+2} = (2n+2) \int_0^{\pi/2} t \sin t \cos^{2n+1} t dt \quad \text{حيث}$$

$$a_{2n+2} = \int_0^{\pi/2} \cos^{2n+2} t dt \quad \text{حيث}$$

$$u(t) = t \quad \begin{cases} u'(t) = 1 \\ v'(t) = -(2n+2) \sin t \cos^{2n+1} t \end{cases} \quad \begin{cases} v(t) = \cos^{2n+2} t \end{cases}$$

$$a_{2n+2} = \left[t \cos^{2n+2} t \right]_0^{\pi/2} + \int_0^{\pi/2} (2n+2) t \sin t \cos^{2n+1} t dt$$

$$= (2n+2) \int_0^{\pi/2} t \sin t \cos^{2n+1} t dt$$

$$L_{2n+1} = V_n + \frac{1}{4} L_n$$

$$V_n = L_{2n+1} - \frac{1}{4} L_n$$

$$\Rightarrow \lim_{n \rightarrow \infty} V_n = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \boxed{\frac{\pi^2}{8}}$$

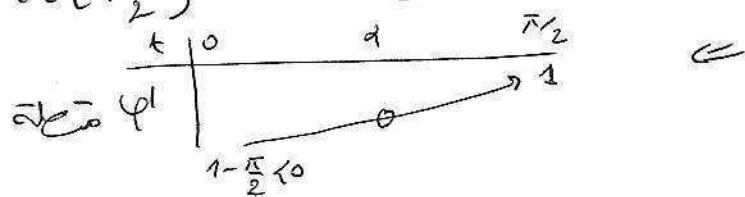
من! نجاز: عبارة 2020 يزيد

$$\psi(t) = t - \frac{\pi}{2} \sin t \quad \text{في } (3)$$

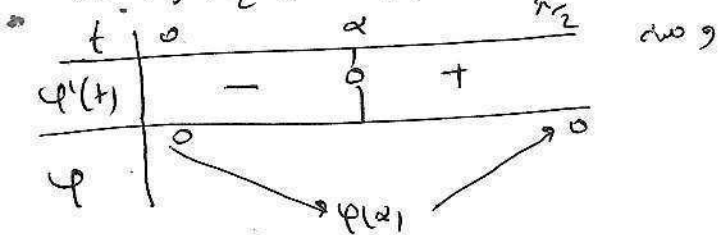
$$t \in [0, \frac{\pi}{2}] \quad \text{و}$$

$$\psi'(t) = 1 - \frac{\pi}{2} \cos t \quad \text{لدينا}$$

$$\forall t \in [0, \frac{\pi}{2}] \quad \psi''(t) = \frac{\pi}{2} \sin t > 0 \quad \text{و}$$



$$\exists! \alpha \in]0, \frac{\pi}{2}[\quad \psi'(\alpha) = 0 \quad \text{و}$$



$$\forall t \in [0, \frac{\pi}{2}] \quad \psi(t) \leq 0 \quad \text{و على}$$

$$\forall t \in [0, \frac{\pi}{2}] : t \leq \frac{\pi}{2} \sin t \quad \text{و بالتالي}$$

$$\frac{1}{(n+1)^2} = 2 \left(\frac{b_n}{a_{2n}} - \frac{b_{n+1}}{a_{2n+2}} \right) \quad \text{على } 9$$

$$L_n = \sum_{k=1}^n \frac{1}{k^2} \quad *$$

$$\frac{1}{k^2} = 2 \left(\frac{b_{k-1}}{a_{2(k-1)}} - \frac{b_k}{a_{2k}} \right) \quad \text{لدينا}$$

$$\sum_{k=1}^n \frac{1}{k^2} = 2 \sum_{k=1}^n \left(\frac{b_{k-1}}{a_{2(k-1)}} - \frac{b_k}{a_{2k}} \right)$$

$$= 2 \left(\frac{b_0}{a_0} - \frac{b_n}{a_{2n}} \right)$$

$$\lim_{n \rightarrow \infty} \frac{b_n}{a_{2n}} = 0 \quad \text{لدينا}$$

$$\frac{b_0}{a_0} = \frac{\frac{\pi^3}{24}}{\frac{\pi}{2}} = \frac{\pi^2}{12}$$

$$\lim_{n \rightarrow \infty} L_n = \boxed{\frac{\pi^2}{6}} \quad \text{على } 9$$

$$V_n = \sum_{k=0}^n \frac{1}{(2k+1)^2}$$

$$L_{2n+1} = \sum_{k=1}^{2n+1} \frac{1}{k^2}$$

$$= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{(2n+1)^2}$$

$$= \left(\frac{1}{1^2} + \frac{1}{3^2} + \dots + \frac{1}{(2n+1)^2} \right)$$

$$+ \left(\frac{1}{2^2} + \frac{1}{4^2} + \dots + \frac{1}{(2n)^2} \right)$$

$$= \sum_{k=0}^n \frac{1}{(2k+1)^2} + \sum_{k=1}^n \frac{1}{(2k)^2}$$

(5)