الجذع المشترك العلمي والتكنولوجي	الحساب المثلثي	سلسلة 2

تمرين 1 : بسط ما يلي :

$$A=\cos^2\left(\frac{\pi}{8}\right)+\cos^2\left(\frac{3\pi}{8}\right)+\cos^2\left(\frac{5\pi}{8}\right)+\cos^2\left(\frac{7\pi}{8}\right)$$

$$B=\sin^2\left(\frac{\pi}{12}\right)+\sin^2\left(\frac{5\pi}{12}\right)+\sin^2\left(\frac{7\pi}{12}\right)+\sin^2\left(\frac{13\pi}{12}\right)$$

$$C=\cos^2\left(\frac{\pi}{12}\right)+\cos^2\left(\frac{11\pi}{12}\right)+2\cos^2\left(\frac{5\pi}{12}\right)$$

$$D=\cos\left(\frac{\pi}{12}\right).\sin\left(\frac{7\pi}{12}\right)+\sin\left(\frac{\pi}{12}\right).\cos\left(\frac{5\pi}{12}\right)$$

$$E=\sin^2\left(\frac{\pi}{12}\right)+\sin^2\left(\frac{3\pi}{12}\right)+\sin^2\left(\frac{5\pi}{12}\right)+\sin^2\left(\frac{7\pi}{12}\right)+\sin^2\left(\frac{9\pi}{12}\right)+\sin^2\left(\frac{11\pi}{12}\right)$$

تمرين 2 : أحسب ما يلي : $\cos\left(\frac{26\pi}{3}\right)$ ، $\tan\left(\frac{85\pi}{4}\right)$ ، $\sin\left(\frac{2015\pi}{6}\right)$

تمرين 3 : علما أن : $\cos\left(\frac{2\pi}{5}\right)=\frac{\sqrt{5}-1}{4}$

1) احسب القيمة المضبوطة لـ $\sin\left(\frac{2\pi}{5}\right)$

2) استنتج القيم المضبوطة لـ $\sin\left(\frac{-2\pi}{5}\right)$ و $\sin\left(\frac{3\pi}{5}\right)$ و $\cos\left(\frac{\pi}{10}\right)$

تمرين 4 : لكل عدد حقيقي x نضع : $A(x)=\sin\left(x+\frac{\pi}{4}\right)+\cos\left(x+\frac{\pi}{4}\right)$

- بين أن : $A(-x)=A(x)$
- بين أن : $A(\pi-x)=-A(x)$

تمرين 5 : x عدد حقيقي ، بين أن :

1) $\sin^4 x-\cos^4 x=\sin^2 x-\cos^2 x$

2) $\sin^4 x+\cos^4 x=1-2\sin^2 x\cos^2 x$

3) $\sin^6 x+\cos^6 x=1-3\sin^2 x\cos^2 x$

4) $\sin^2\left(\frac{2\pi}{5}\right)-\cos^2\left(\frac{2\pi}{5}\right)=\cos^2\left(\frac{\pi}{10}\right)-\sin^2\left(\frac{\pi}{10}\right)$

تمرين 1 :

$$=\cos^2\left(\frac{\pi}{8}\right)+\cos^2\left(\frac{3\pi}{8}\right)+\cos^2\left(\frac{5\pi}{8}\right)+\cos^2\left(\frac{7\pi}{8}\right)$$

$$A=\cos^2\left(\frac{\pi}{8}\right)+\cos^2\left(\frac{3\pi}{8}\right)+\cos^2\left(\pi-\frac{3\pi}{8}\right)+\cos^2\left(\pi-\frac{\pi}{8}\right)$$

$$A=\cos^2\left(\frac{\pi}{8}\right)+\cos^2\left(\frac{3\pi}{8}\right)+\left(-\cos\left(\frac{3\pi}{8}\right)\right)^2+\left(-\cos\left(\frac{\pi}{8}\right)\right)^2$$

$$A=\cos^2\left(\frac{\pi}{8}\right)+\cos^2\left(\frac{3\pi}{8}\right)+\cos^2\left(\frac{3\pi}{8}\right)+\cos^2\left(\frac{\pi}{8}\right)$$

$$A=2\left(\cos^2\frac{\pi}{8}+\cos^2\frac{3\pi}{8}\right)=2\left(\cos^2\frac{\pi}{8}+\cos^2\left(\frac{\pi}{2}-\frac{\pi}{8}\right)\right)$$

$$A=2\left(\cos^2\frac{\pi}{8}+\sin^2\frac{\pi}{8}\right)=2$$

$$B=\sin^2\left(\frac{\pi}{12}\right)+\sin^2\left(\frac{5\pi}{12}\right)+\sin^2\left(\frac{7\pi}{12}\right)+\sin^2\left(\frac{13\pi}{12}\right)$$

$$B=\sin^2\left(\frac{\pi}{12}\right)+\sin^2\left(\frac{\pi}{2}-\frac{\pi}{12}\right)+\sin^2\left(\frac{7\pi}{12}\right)+\sin^2\left(\frac{\pi}{2}+\frac{7\pi}{12}\right)$$

$$B=\sin^2\left(\frac{\pi}{12}\right)+\cos^2\left(\frac{\pi}{12}\right)+\sin^2\left(\frac{7\pi}{12}\right)+\cos^2\left(\frac{7\pi}{12}\right)$$

$$B=1+1=2$$

$$C=\cos^2\left(\frac{\pi}{12}\right)+\cos^2\left(\frac{11\pi}{12}\right)+2\cos^2\left(\frac{5\pi}{12}\right)$$

$$C=\cos^2\left(\frac{\pi}{12}\right)+\cos^2\left(\frac{5\pi}{12}\right)+\cos^2\left(\frac{5\pi}{12}\right)+\cos^2\left(\frac{11\pi}{12}\right)$$

$$C=\cos^2\left(\frac{\pi}{12}\right)+\cos^2\left(\frac{\pi}{2}-\frac{\pi}{12}\right)+\cos^2\left(\frac{5\pi}{12}\right)+\cos^2\left(\frac{\pi}{2}+\frac{5\pi}{12}\right)$$

$$C=\cos^2\left(\frac{\pi}{12}\right)+\sin^2\left(\frac{5\pi}{12}\right)+\cos^2\left(\frac{5\pi}{12}\right)+\left(-\sin\left(\frac{5\pi}{12}\right)\right)^2$$

$$C=1+\cos^2\left(\frac{5\pi}{12}\right)+\sin^2\left(\frac{5\pi}{12}\right)=1+1=2$$

$$D=\cos\left(\frac{\pi}{12}\right).\sin\left(\frac{7\pi}{12}\right)+\sin\left(\frac{\pi}{12}\right).\cos\left(\frac{5\pi}{12}\right)$$

$$D=\cos\left(\frac{\pi}{12}\right).\sin\left(\frac{\pi}{2}+\frac{\pi}{12}\right)+\sin\left(\frac{\pi}{12}\right).\cos\left(\frac{\pi}{2}-\frac{\pi}{12}\right)$$

$$D=\cos\left(\frac{\pi}{12}\right).\cos\left(\frac{\pi}{12}\right)+\sin\left(\frac{\pi}{12}\right).\sin\left(\frac{\pi}{12}\right)$$

$$D=1$$

$$E=\sin^2\left(\frac{\pi}{12}\right)+\sin^2\left(\frac{3\pi}{12}\right)+\sin^2\left(\frac{5\pi}{12}\right)+\sin^2\left(\frac{7\pi}{12}\right)+\sin^2\left(\frac{9\pi}{12}\right)+\sin^2\left(\frac{11\pi}{12}\right)$$

$$E=\sin^2\left(\frac{\pi}{12}\right)+\sin^2\left(\frac{7\pi}{12}\right)+\sin^2\left(\frac{3\pi}{12}\right)+\sin^2\left(\frac{9\pi}{12}\right)+\sin^2\left(\frac{5\pi}{12}\right)+\sin^2\left(\frac{11\pi}{12}\right)$$

$$E=\sin^2\left(\frac{\pi}{12}\right)+\sin^2\left(\frac{\pi}{2}+\frac{\pi}{12}\right)+\sin^2\left(\frac{3\pi}{12}\right)+\sin^2\left(\frac{\pi}{2}+\frac{3\pi}{12}\right)+\sin^2\left(\frac{5\pi}{12}\right)+\sin^2\left(\frac{\pi}{2}+\frac{5\pi}{12}\right)$$

$$E=\sin^2\left(\frac{\pi}{12}\right)+\cos^2\left(\frac{\pi}{12}\right)+\sin^2\left(\frac{3\pi}{12}\right)+\cos^2\left(\frac{3\pi}{12}\right)+\sin^2\left(\frac{5\pi}{12}\right)+\cos^2\left(\frac{5\pi}{12}\right)$$

$$E=3$$

من خلال هذه الأمثلة ستلاحظ أن تبسيط مثل هذه التعابير يعتمد على أمرين هامين :

- أولا ملاحظة العلاقة بين الأعداد الموجودة مثلا، إذا اعتبرنا العددين $\frac{\pi}{12}$ و $\frac{5\pi}{12}$ فبعد جمعهما نجد: $\frac{\pi}{2}$ أي أن: $\frac{5\pi}{12}=\frac{\pi}{2}-\frac{\pi}{2}$

بينما إذا اعتبرنا العددين: $\frac{\pi}{7}$ و $\frac{8\pi}{7}$ فبعد طرحهما نجد: π أي أن: $\frac{8\pi}{7}=\pi+\frac{\pi}{7}$

- الأمر الثاني هو استعمال هذه الملاحظة و تطبيق قواعد الحساب المثلثي لأجل التبسيط و الحساب إن أمكن.

تمرين 2 :

$$\cos\left(\frac{26\pi}{3}\right)=\cos\left(\frac{24\pi+2\pi}{3}\right)=\cos\left(8\pi+\frac{2\pi}{3}\right)=\cos\left(\frac{2\pi}{3}\right)=\frac{-1}{2}$$

$$\tan\left(\frac{85\pi}{4}\right)=\tan\left(\frac{84\pi+\pi}{4}\right)=\tan\left(21\pi+\frac{\pi}{4}\right)=\tan\left(20\pi+\pi+\frac{\pi}{4}\right)=\tan\left(\pi+\frac{\pi}{4}\right)=\tan\left(\frac{\pi}{4}\right)=1$$

$$\sin\left(\frac{71\pi}{3}\right)=\sin\left(\frac{72\pi-\pi}{3}\right)=\sin\left(24\pi-\frac{\pi}{3}\right)=\sin\left(-\frac{\pi}{3}\right)=-\sin\left(\frac{\pi}{3}\right)=-\frac{\sqrt{2}}{2}$$

قد نتساءل عن طريقة كتابة الأعداد أعلاه

الأمر ببساطة يعتمد على إجراء قسمة عادية ، مثلا بعد قسمة 26 على 3 سيكون الخارج 8 و الباقي 2 لذلك كتبنا $26\pi=3\times 8\pi+2\pi=24\pi+2\pi$ ، ويمكن أيضا كتابته على شكل فرق: $26\pi=27\pi-\pi$ ، لكن يستحسن الحل الأول لأنه يعطينا عددا زوجيا في الخارج بعد القسمة مما يسمح بتطبيق مباشر للخاصية: $\sin(x+2k\pi)=\sin(x)$ أو $\cos(x+2k\pi)=\cos(x)$ أما بالنسبة لـ $\tan$ فالأمر ليس ضروريا لأن: $\tan(x+k\pi)=\tan(x)$

تمرين 3 :

نعلم أن: $\sin^2\left(\frac{2\pi}{5}\right)+\cos^2\left(\frac{2\pi}{5}\right)=1$ منه: $\sin^2\left(\frac{2\pi}{5}\right)+\left(\frac{\sqrt{5}-1}{4}\right)^2=1$ منه: $\sin^2\left(\frac{2\pi}{5}\right)+\frac{6-2\sqrt{5}}{16}=1$

منه: $\sin^2\left(\frac{2\pi}{5}\right)=1-\frac{6-2\sqrt{5}}{16}=\frac{16-6+2\sqrt{5}}{16}=\frac{10+2\sqrt{5}}{16}$

وبما أن : $\frac{2\pi}{5}\in[0;\pi]$ فإن: $\sin\left(\frac{2\pi}{5}\right)\geq 0$ ، بالتالي: $\boxed{\sin\left(\frac{2\pi}{5}\right)=\frac{\sqrt{10+2\sqrt{5}}}{4}}$

$\boxed{\sin\left(\frac{-2\pi}{5}\right)=-\sin\left(\frac{2\pi}{5}\right)=-\frac{\sqrt{10+2\sqrt{5}}}{4}}$ ، $\boxed{\sin\left(\frac{3\pi}{5}\right)=\sin\left(\pi-\frac{2\pi}{5}\right)=\sin\left(\frac{2\pi}{5}\right)=\frac{\sqrt{10+2\sqrt{5}}}{4}}$

$$\boxed{\cos\left(\frac{\pi}{10}\right)=\cos\left(\frac{\pi}{2}-\frac{2\pi}{5}\right)=\sin\left(\frac{2\pi}{5}\right)=\frac{\sqrt{10+2\sqrt{5}}}{4}}$$

تذكر دائما القاعدة الأساسية للحساب المثلثي: $\sin^2(x)+\cos^2(x)=1$

تمرين 4 : $A(x)=\sin\left(x+\frac{\pi}{4}\right)+\cos\left(x+\frac{\pi}{4}\right)$

1

لدينا :

$$A(-x)=\sin\left(-x+\frac{\pi}{4}\right)+\cos\left(-x+\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{2}-\left(-x+\frac{\pi}{4}\right)\right)+\sin\left(\frac{\pi}{2}-\left(-x+\frac{\pi}{4}\right)\right)$$

$$A(-x)=\cos\left(\frac{\pi}{2}+x-\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{2}+x-\frac{\pi}{4}\right)=\cos\left(x+\frac{\pi}{4}\right)+\sin\left(x+\frac{\pi}{4}\right)=A(x)$$

2

لدينا :

$$A(\pi-x)=\sin\left(\pi-x+\frac{\pi}{4}\right)+\cos\left(\pi-x+\frac{\pi}{4}\right)=\sin\left(\frac{\pi}{2}-\left(\pi-x+\frac{\pi}{4}\right)\right)+\cos\left(\left(\frac{\pi}{2}-\left(\pi-x+\frac{\pi}{4}\right)\right)\right)$$

$$A(\pi-x)=\cos\left(\frac{\pi}{2}-\pi+x-\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{2}-\pi+x-\frac{\pi}{4}\right)=\cos\left(x+\frac{\pi}{4}-\pi\right)+\sin\left(x+\frac{\pi}{4}-\pi\right)$$

$$A(\pi-x)=\cos\left(x+\frac{\pi}{4}+\pi\right)+\sin\left(x+\frac{\pi}{4}+\pi\right)=-\cos\left(x+\frac{\pi}{4}\right)-\sin\left(x+\frac{\pi}{4}\right)=-A(x)$$

في كلا السؤالين استعملنا القواعد التالية : $\sin(x)=\cos\left(\frac{\pi}{2}-x\right)$ و $\cos(x)=\sin\left(\frac{\pi}{2}-x\right)$ و

$\sin(x-\pi)=\sin(x+\pi)=-\sin(x)$ و $\cos(x-\pi)=\cos(x+\pi)=-\cos(x)$

تمرين 5 : x عدد حقيقي

1

$$\sin^4 x-\cos^4 x=\left(\sin^2 x\right)^2-\left(\cos^2 x\right)^2=\left(\sin^2 x+\cos^2 x\right)\left(\sin^2 x-\cos^2 x\right)=1\times\left(\sin^2 x-\cos^2 x\right)$$

$$\sin^4 x-\cos^4 x=\sin^2 x-\cos^2 x$$

2

$$\sin^4 x+\cos^4 x=\sin^4 x+2\sin^2 x\cos^2 x+\cos^4 x-2\sin^2 x\cos^2 x$$

$$\sin^4 x+\cos^4 x=\left(\sin^2 x+\cos^2 x\right)^2-2\sin^2 x\cos^2 x=1-2\sin^2 x\cos^2 x$$

3

$$\sin^6 x+\cos^6 x=\left(\sin^2 x\right)^3+\left(\cos^2 x\right)^3=\left(\sin^2 x+\cos^2 x\right)\left(\sin^4 x-\sin^2 x\cos^2 x+\cos^2 x\right)$$

$$\sin^6 x+\cos^6 x=1\times\left(\sin^4 x+2\sin^2 x\cos^2 x+\cos^2 x-3\sin^2 x\cos^2 x\right)$$

$$\sin^6 x+\cos^6 x=\left(\sin^2 x+\cos^2 x\right)^2-3\sin^2 x\cos^2 x$$

$$\sin^6 x+\cos^6 x=1-3\sin^2 x\cos^2 x$$

4

$$\sin^2\left(\frac{2\pi}{5}\right)-\cos^2\left(\frac{2\pi}{5}\right)=\cos^2\left(\frac{\pi}{2}-\frac{2\pi}{5}\right)-\sin^2\left(\frac{\pi}{2}-\frac{2\pi}{5}\right)=\cos^2\left(\frac{\pi}{10}\right)-\sin^2\left(\frac{\pi}{10}\right)$$